$\frac{f_1 y}{c} z\,dy$

$\frac{f_1}{c} \int z y^2\,dy = \frac{f_1}{c}$

$\frac{f_1 - f_2}{c} A\bar{y} = \text{max } a\ x$

$\frac{f_1}{c} I = M$

$\frac{f_2}{c} I = M + dM$

$\frac{f_1 - f_2}{c} I = dM$

$\frac{f_1 - f_2}{c} = \frac{dM}{I}$

$\frac{dM}{I} A\bar{y} = \text{Alpha } x$

APPLIED MECHANICS.

PART I.

BY

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CONTENTS.

CHAPTER I.

DEFINITIONS AND GENERAL PRINCIPLES.

CHAPTER A.

CHAPTER II.

THE EQUILIBRIUM AND STRENGTH OF BEAMS.

CHAPTER B.

THE EQUILIBRIUM AND STRENGTH OF BEAMS.

CHAPTER III.

TRANSVERSE STRENGTH OF BEAMS.

CHAPTER IV.

PILLARS.

CHAPTER C.

DEFLECTION OF STRUTS.

CHAPTER V.

TORSION.

CHAPTER VI.

STRENGTH OF HOLLOW CYLINDERS AND SPHERES.

APPLIED MECHANICS.

CHAPTER I.

DEFINITIONS AND GENERAL PRINCIPLES.

(1).—*Definitions.*—The science relating to the strength of materials is partly theoretical, partly practical. Its primary object is to investigate the forces developed within a body, and to determine the most economical dimensions and form, consistent with stability, of that body. Certain hypotheses have to be made, but they are of such a nature as always to be in accord with the results of direct observation.

The materials in ordinary use for structural purposes may be termed, generally, *solid bodies*, *i.e.*, bodies which offer an appreciable resistance to a change of form.

A body acted upon by external forces is said to be *strained* or *deformed*, and the straining or deformation induces *stress* amongst the particles of the body.

The state of strain is *simple* when the stress acts in *one* direction only, and the strain itself is measured by the ratio of the deformation to the original length.

The state of strain is *compound* when *two* (or *more*) stresses act simultaneously in different directions.

A strained body tends to assume its natural state when the straining forces are removed: this tendency is called its *elasticity*. A thorough knowledge of the laws of elasticity, *i. e.*, of the laws which connect the external forces with the internal stresses, is absolutely necessary for the proper comprehension of the strength of materials. This property of elasticity is not possessed to the same degree by all bodies. It may be almost absolute, or almost zero, but in the majority of cases it has a mean value. Hence it naturally follows that solid bodies may be classified between two extreme, though ideal, states, viz. :—a *perfectly elastic* state and a *perfectly soft* state. Perfectly elastic bodies which have been strained, resume their original forms exactly when the straining forces are removed. Perfectly soft bodies are wholly devoid of elasticity, and offer no resistance to a change of form.

(2).–*Stresses and Fracture.*–Every body may be subjected to 5 distinct stresses:–

(*a*).–A longitudinal pull, or tension.

(*b*).–A longitudinal thrust, or compression.

(*c*).–A shear, or tangential stress, which may be defined as a stress tending to make one surface slide over another with which it is in contact.

(*d*).–A transverse stress.

(*e*).–A twist or torsion.

If any one of these stresses exceed a certain limit, fracture ensues.

(3).–*Resistance of bars to tension and compression.*–Let a straight bar of length L be stretched or compressed longitudinally by a force P uniformly distributed over the constant cross-section A of the bar; and let l be the consequent extension or compression, *i. e.*, the deformation.

Then, if the transverse dimensions are small compared with the length, experiment shews that, *within certain limits*, the force P is *directly* proportional to the deformation l, and to the area A, and *inversely* proportional to the length L, these quantities being connected by the relation,

$$P = E.A.\frac{l}{L}$$

where E is some constant dependent upon the material of the bar.

This constant is called the *co-efficient of elasticity.*

If A is unity, and if the bar is stretched until $l = L$, then E becomes equal to P; thus E is the force that will double the length of a bar of which the sectional area is uniform and equal to unity.

The equation is the analytical expression of Hooke's Law for a body in a state of simple strain, viz., that the strain is proportional to the stress, for it may be written, $\frac{P}{A} = E.\frac{l}{L}$, shewing that the unit stress is E times the unit strain.

(4).–*Specific Weight; Co-efficient of Elasticity; Limit of Elasticity; Breaking Weight.*—Before the strength of a body can be fully known, it is necessary to obtain the values of four constants. These constants depend upon the nature of the material, and are:–the *specific weight*, the *co-efficient of elasticity*, the "*limit of elasticity*," and the *breaking weight* (or *stress*).

(*a*).–*Specific Weight.*–The specific weight is the weight of a unit of volume. The specific weights of most of the materials of construction

have been carefully determined and tabulated. If the specific weight of any new material is required, a convenient approximate method is to prepare from it a number of regular solids of determinate volume, and weigh them in an ordinary pair of scales. The ratio of the total weight of these solids to their total volume is the specific weight. It must be remembered that the weight may vary considerably with time, etc.; thus, a sample of greenheart weighed 69.75-lbs. per cubic foot when first cut out of the log, and only 57-lbs. per cubic foot at the end of six months.

The *total load* upon a structure includes *all* the external forces applied to it, and in practice is designated *dead* (*permanent*) or *live* (*rolling*), according as the forces are gradually applied and steady, or suddenly applied and accompanied with vibrations. For example, the weight of a bridge is a dead load, while a train passing over it is a live load; the weight of a roof together with the weight of any snow which may have accumulated upon it, is a dead load; *wind* causes at times excessive vibrations in the members of a structure, and although often treated as a dead load, should in reality be considered a live load.

The dead loads of many structures (as masonry walls, etc.) are so great that extra or accidental loads may be safely disregarded. In cold climates, great masses of snow, and the penetrating effect of the frost, necessitate very deep foundations, which proportionately increase the dead weight.

(*b*).–*Co-efficient of Elasticity.*–Generally speaking, a knowledge of the external forces acting upon a structure discloses the manner of their distribution amongst its various members, but the deformation of these members can only be estimated by means of the co-efficient of elasticity, which expresses the relation between a stress and the corresponding strain.

In a homogeneous solid there may be 21 distinct co-efficients of elasticity, which are usually classified under the following heads:–

(1).–*Direct*, expressing the relation between longitudinal strains and normal stresses in the same direction.

(2).–*Transverse*, expressing the relation between tangential stresses and strains in the same direction.

(3).–*Lateral*, expressing the relation between longitudinal strains and normal stresses at right angles to the strains, *i. e.*, a lateral resistance to deformation.

(4).–*Oblique*, expressing other relations of stress and strain.

If a body is *isotropic, i. e.*, equally elastic in all directions, the 21 co-efficients reduce to 2, viz., the co-efficient of direct elasticity and the co-efficient of lateral elasticity. Such bodies, however, are almost wholly ideal.

Co-efficients of elasticity must be determined by experiment.

The co-efficients of direct elasticity for the different metals and timbers are sometimes obtained by subjecting bars of the material to forces of extension or compression, but generally by observing the deflections of beams loaded transversally. The co-efficients for blocks of stone and masonry might also be found by transverse loading; they are of little, if any, practical use, for on account of the inherent stiffness of masonry structures, their deformations, or *settlings*, are due rather to defective workmanship than to the natural play of elastic forces.

The *torsional* co-efficient of elasticity, *i. e.*, the co-efficient of elastic resistance to torsion, has been shewn by experiment to vary from 2-5ths to 3-8ths of the co-efficient of direct elasticity.

(*c*).–*Limit of Elasticity.*–When the forces which strain a body fall below a certain limit, the body, on the removal of the forces, will resume its original form and dimensions without sensible change, and may be treated as perfectly elastic. But if the forces exceed this limit, the body will receive a permanent deformation, or, as it is termed, a *set.*

Such a limit is called a *limit of elasticity*, and is the greatest stress that can be applied to a body without producing in it an appreciable and permanent deformation.

For example:–

A bar of average iron, one sq. in. in section, will stretch

$\frac{1}{13,000}$ part of its original length, under a weight of 2000-lbs.

$\frac{2}{13,000}$ " " " 4000 "

$\frac{3}{13,000}$ " " " 6000 "

and so on.

This stretching increases uniformly until the weight is about 24,000 lbs., and is of such a character that the bar, when relieved from the several weights, returns to its original length. If the bar is then strained by a greater weight than 24,000-lbs. it receives a set, so that 24,000-lbs. per sq. in. is the elastic limit of the iron of which the bar is composed. Similar reasoning holds true for forces of compression.

Some authorities hold that the application to a body of any stress, however small, introduces a permanent set. Even if this be so, the deformations under loads which are less than the elastic limit are so slight as to be of no practical account, and may be safely disregarded. Probably they are only temporary, or *false*, and may be made to disappear by freeing the body from strain and allowing it to rest; the set will become permanent if the body remains loaded for some considerable time.

The main object, then, of the *theory* of the strength of materials is to determine whether the stresses developed within any particular member of a structure exceed the limit of elasticity. As soon as they do so, that member is permanently deformed, its strength is impaired, it becomes predisposed to rupture, and the safety of the whole structure is threatened. Still, it must be borne in mind, that it is not absolutely true that a material is always weakened by being subjected to forces superior to this limit. In the manufacture of iron bars, for instance, each of the processes through which the metal passes, changes its elasticity, and increases its strength. Such a material is to be treated as being in a new state and as possessing new properties.

The essential requirement of practice is a tough material with a high elastic limit. This is especially necessary for bridges and all structures liable to constantly repeated loads, for it is found that these repetitions lower the elastic limit and diminish the strength.

In the majority of cases experience has fixed a practical limit for the stresses, much below the limit of elasticity. This ensures greater safety and provides against unforeseen and accidental loads, which may exceed the *practical* limit, but which do no harm unless they pass the *elastic* limit.

Certain operations have the effect of raising the limit of elasticity: a wrought iron bar, steadily strained almost to the point of its ultimate strength and then released from strain and allowed to rest, experiences an elevation both of tenacity and of the elastic limit. A similar result follows, in the various processes employed in the manufacture of iron and steel bars and wire. Again, iron and steel bars subjected to *long-continued* compression or extension have their resistance increased, mainly because time is allowed for the molecules of that metal to assume such positions as will enable them to offer the maximum resistance; the increase is not attended by any appreciable change of density.

(*d*).–*Breaking Weight.*–If the external forces which act upon a body, increase indefinitely, the limit of elasticity is soon passed, the deformation

increases, and at last a stress is developed which produces fracture. This final stress, which may be a pull, a pressure, a shear, a bending stress, or the result of a twist, is called the *breaking weight*, and defines the ultimate strength of the body with respect to the kind of stress in question. The fracture of a material is important from two points of view :–

(1).–It discloses the treatment to which the material should be subjected.

(2).–It indicates the properties of the materials.

Numerous experiments have been made to determine the breaking weights of the ordinary materials of construction, but their properties are so variable as to necessitate fresh experiments in almost every new structure.

Cast iron is, perhaps, the most doubtful of all materials, and the greatest care should be observed in its employment. It possesses little tenacity or elasticity, is very hard and brittle, and may fail suddenly under a shock, or an extreme variation of temperature. Unequal cooling may predispose the metal to rupture, and its strength may be still further diminished by the presence of air-holes.

Wrought iron and steel are far more uniform in their behaviour, and obey with tolerable regularity certain theoretical laws. They are tenacious, ductile, have great compressive strength, and are most reliable for structural purposes. Their strength and elasticity may be considerably reduced by high temperatures or severe cold.

From recent researches as to the tenacity of wrought iron and steel bars it has been inferred that the test pieces should be :–

(1).–Cylindrical in form.

(2).–At least ½ inch in diameter.

(3).–At least 4 diameters in length, and 5 or 6 diameters if the metal is very soft and ductile.

Timber is usually tested by tension, compression, and transverse loading.

The chief object of experiments upon *masonry* and *brickwork* is to discover their resistance to compression, *i.e.*, their crushing strength. In fact, their stiffness is so great that they may be compressed up to the point of fracture without sensible change of form, and it is therefore very difficult, if not impossible, to observe the limit of elasticity.

The *cement* or *mortar* uniting the stones and bricks is most irregular in quality. In every important work it should be an invariable rule to prepare specimens for testing. The crushing strength of cement and mortar is much greater than the tensile strength, the latter being often

exceedingly small. Hence it is advisable to avoid tensile stresses within a mass of masonry, as they tend to open the joints and separate the stones from one another. Attempts are frequently made to strengthen masonry and brickwork walls by inserting in the joints tarred and sanded strips of hoop-iron. Their utility is doubtful, for, unless well protected from the atmosphere, they oxidize, to the detriment of the surrounding material, and, besides this, they prevent an equable distribution of pressure. They are, however, far preferable to bond timbers.

(5).–*Working Load; Factor of Safety.*–The *working load* is the heaviest dead load that can be applied in practice, and is only a small proportion of the breaking weight (or load). The ratio of the breaking weight to the working load is called a *factor of safety*, and its value depends both upon the nature of the load and upon the nature of the material. The effect of a live load is almost *twice* as great as that of a dead load of equal weight, and the former therefore requires a proportionately larger factor of safety. If the load is *mixed, i.e.*, partly dead and partly live, the live portion, if multiplied by 2, is converted into a dead load, and the same factor of safety may then apply to the whole, or else a compound factor of safety may be deduced as follows:–

Let w_1, w_2 be respectively the dead and live loads
" f_1, f_2 " corresponding factors of safety.

$$\therefore \text{ the compound factor of safety} = \frac{f_1 . w_1 + f_2 . w_2}{w_1 + w_2}.$$

Again, a larger factor of safety is required for a suddenly than for a gradually applied load, for an unreliable than for a reliable material, for a member of a structure exposed to different stresses than for one uniformly strained.

(6).–*Proof Load.*–When a material is to be employed in construction it is usual to *prove* it, by subjecting pieces of the material to a *proof load*, which will produce a *proof stress* and a *proof strain* sufficient to reveal any defects. If the pieces themselves are to be afterwards used the proof stress should not exceed the limit of elasticity, and is generally about double the working stress.

(7).–*Fractured area; Tensile limit.*–Absolute strength ought not to be the sole qualification of a material. An iron or a steel may be very strong, but if it is not ductile, it will be hard and brittle, and unsuited to withstand a shock, while, on the other hand, it may be too ductile to be of service. The ductility diminishes the ultimate strength but increases the working strength, so that a soft metal, being less liable to

snap or break suddenly, is well adapted to positions subjected to vibration and concussion. Hence will be seen the importance of expressing the tensile strength of the test-piece by so many pounds per sq. in. of the *fractured* area, instead of taking the original sectional area as the standard of reference. Probably it would be still more scientific and correct to observe the change of form at the instant the metal ceases to resist an increase of stress. At this point, which has been designated the *tensile limit*, experiments shew that the reduction of the sectional area is about one-half, and the elongation is about three-fourths of what these would respectively be at the point of fracture. If, however, the fracture is caused by a *sudden* stress, the fractured dimensions seem almost identical with those at the tensile limit, had the metal been *steadily* strained up to this latter point.

(8).—*Resilience, or Spring.*—Let it be required to determine the *work done* in stretching or compressing a bar of length L and sectional area A, by an amount l.

Suppose that the force applied to the bar gradually increases from 0 until it attains the value P; its mean value is $\frac{P}{2}$; and the *work done* is therefore $\frac{P}{2}.\,l$.

But $P=E\,A.\,\frac{l}{L}$, (E being the co-efficient of elasticity.)

$$\therefore \text{ the work done} = \frac{E}{2}.A.\frac{l^2}{L} = \frac{1}{E}.\left(\frac{P}{A}\right)^2.\frac{AL}{2}.$$

This formula is only true for small values of the ratio $\frac{l^2}{L}$; in the case of a compressive force it is assumed that the bar does not bend.

A *suddenly* applied force, $\frac{P}{2}$, will do as much work as a *steady* force which increases uniformly from 0 to P, and hence it follows that a bar requires *twice* the strength to resist with safety the sudden application of a given load than is necessary when the same load is gradually applied.

If f is the *proof stress* per unit of sectional area, $\frac{f}{E}$ is the corresponding *proof strain*, and the work done in producing the latter is called the *resilience* of the bar. According to the above, its value is $\frac{f^2}{E}.\frac{A.L}{2}$; $\frac{f^2}{E}$ is called the Modulus of Resilience.

Ex. 1. A wrought-iron tie-rod, 30-ft. in length and 4 sq.-ins. in sectional area, is subjected to a longitudinal pull of 40,000-lbs. Determine the unit stress, the strain, and the elongation, the co-efficient of elasticity being 30,000,000-lbs.

$$\text{The unit stress is } \frac{40{,}000}{4} = 10{,}000\text{-lbs. per sq. in.}$$

Also, from the elastic law, $10{,}000 = 30{,}000{,}000 \times \text{strain}$.

$$\therefore \text{ the strain} = \frac{1}{3000}$$

$$\text{and the elongation} = \frac{30}{3000} = \frac{1}{100}\text{-ft.}$$

Ex. 2. A steel rod is 15-ft. long, and $2\frac{1}{2}$ sq.ins. in sectional area. The proof strain of the steel is $\frac{1}{1000}$, and its co-efficient of elasticity is 36,000,000 lbs. Find the greatest weight that can be safely allowed to fall upon the end of the rod from a height of 27-ft.

The proof stress $= E \times$ proof strain $= 36{,}000$-lbs. per sq. in.

The compression of the rod under the proof stress is $\frac{15}{1000} = \frac{3}{200}$-ft.

The resilience of the rod $= \frac{f^2}{E} \cdot \frac{A.L}{2} = \frac{(36{,}000)^2}{36{,}000{,}000} \cdot \frac{2\frac{1}{2} \times 15 \times 12}{2}$

$= 8100$ *inch*-lbs. $= 675$ ft.-lbs.

Again, let W be the required weight in lbs.

The total distance through which it falls $=$ 27-ft. $+$ compression $= \left(27 + \frac{3}{200}\right)$ ft., and the corresponding work is $W.\left(27 + \frac{3}{200}\right)$ ft.-lbs. This must of course be exactly equivalent to the resilience of the rod,

$$\text{and } \therefore W.\left(27 + \frac{3}{200}\right) = 675$$

$$\therefore W = 24.9 \text{ lbs.}$$

Note.—The resilience of the rod may also be at once found from the fact that it is the product of one-half of the total stress by the compression, *i.e.*, $\frac{1}{2} \cdot 2\frac{1}{2} \cdot 36{,}000 \times \frac{3}{200} = 675$ ft.-lbs.

(9).—*Wöhler's Law.*—It is now generally admitted that variable forces, constantly repeated loads, and continued vibrations, diminish the strength of a material, whether they produce stresses approximating to the elastic limit, or exceedingly small stresses occurring with great rapidity. Indeed many engineers design structures in such a manner

that the several members are strained in one way only, so convinced are they of the evil effect of alternating tensile and compressive stresses. Although the fact of a variable ultimate strength had thus been tacitly acknowledged and often allowed for, Wöhler was the first to give formal expression to it and, as a result of observation and experiment, deduced the following law:–

"That if a stress t, due to a static load, cause the fracture of a bar, "the bar may also be fractured by a series of often repeated stresses, "each of which is less than t; and that, as the differences of stress "increase, the cohesion of the material is affected in such a manner "that the minimum stress required to produce fracture is diminished."

This law is manifestly incomplete. In Wöhler's experiments the applications of the load followed each other with great rapidity, yet a certain time was required for the resulting stresses to attain their full intensity; the influence due to the rapidity of application, to the rate of increase of the stress, and to the duration of individual strains, still remains a subject for investigation.

From the law, however, as it stands, formulæ may be deduced which, it is claimed, are more in accordance with the results of experiment, give smaller errors, and ensure greater safety, than the false assumption of a constant ultimate strength.

The formulæ necessarily depend upon certain experimental results, but in applying them to any particular case it must be remembered, that only such results should be employed as have been obtained for material of the same kind, and under the same conditions as the material under consideration. The effects due to faulty material, rust, etc., are altogether indeterminate, so that no formula can be perfectly universal in its application. Hence the necessity for factors of safety with values depending upon the class of structure still exists.

A brief description of the principal of these formulæ will now be given and in the discussion,

t, *the statical breaking strength*, is the resistance to fracture under a static, or under a very gradually applied load.

u, *the primitive strength*, is the resistance to fracture under a given number of repeated stresses, the stress in each repetition remaining unchanged in kind, *i. e.*, being due either to a tension, a compression, or a shear.

s, *the vibration strength*, is the resistance to fracture under alternating stresses of equal intensities, but different in kind, due to a vibratory motion about the unstrained state of equilibrium.

b is the admissible stress per unit of sectional area.

F is the effective sectional area, and is

$$=\frac{\textit{numerically absolute maximum load}}{b}$$

(10).–*Launhardt's formula.*–A bar of unit sectional area is subjected to stresses (B) which are either wholly tensile, wholly compressive, or wholly shearing, and which vary from a maximum a_1 (=max.B) to a minimum a_2 (=min.B).

Let $a_1 - a_2 = d =$ the maximum difference of stress.

$$\text{Let} \frac{a_2}{a_1} = \frac{min.\ B}{max.\ B} = \phi$$

$$\text{If } a_2=0, \ \therefore\ a_1=d=u$$

$$\text{If } d=0, \ \therefore\ a_1=a_2=t$$

By Wöhler's law,

$$a_1 \propto d = f.\ d \qquad (1)$$

f being an unknown co-efficient of which the value remains to be determined.

$$\text{If } d=0, \ \therefore a_1=t \text{ and } f=\infty$$

$$\text{If } d=u, \ \therefore a_1=d \text{ and } f=1$$

Launhardt's assumption, viz., $f = \dfrac{t-u}{t-a_1}$, satisfies these extreme conditions, and also gives intermediate values to a_1 which closely agree with the results of the most reliable experiments.

Hence, (1) becomes, $a_1 = \dfrac{t-u}{t-a_1}.\ d = \dfrac{t-u}{t-a_1}.\ (a_1 - a_2)$

$$\text{and } \therefore\ a_1 = u.\left(1 + \frac{t-u}{u}.\frac{a_2}{a_1}\right) = u.\left(1 + \frac{t-u}{u}.\ \phi\right) \qquad (2)$$

This is Launhardt's formula, and is the expression of Wöhler's Law.

Wöhler in his bending experiments upon Phœnix axle-iron found that

* $u=2195^k$ per cent.^{2} and $t=4020^k$ per cent.^{2}; $\therefore \dfrac{t-u}{u} = \dfrac{5}{6}$

The same iron under tension gave $u=2195^k$ per cent.^{2} and $t=3290^k$ per cent.^{2}, $\therefore \dfrac{t-u}{u} = \dfrac{1}{2}$.

Choosing the most unfavourable case, and, in order to ensure greater safety, taking $u=2100^k$ per cent.^{2}, equation (2) becomes

$$a_1 = 2100.\left(1 + \frac{\varphi}{2}\right) \qquad (3)$$

---

* k per cent.^{2} is an abbrevation for kilogrammes per square centimetre.
1 kilogramme per sq. centimetre is equivalent to 14.2232-lbs. per sq. in.

If 3 is the factor of safety,

$$\therefore\ b=700.\left(1+\frac{\phi}{2}\right) \tag{4}$$

In his bending experiments upon Krupp cast steel (untempered) it was found that $u=3510^k$ per cent.[2] and $t=7340^k$ per cent.[2], $\therefore \frac{t-u}{u}=\frac{7}{6}$.

But steel varies considerably in strength, and great care must be exercised in its use, especially in bridge construction. For this reason take $u=3300^k$ per cent.[2] and $t=6000^k$ per cent.[2], $\therefore \frac{t-u}{u}=\frac{9}{11}$ and (2) becomes,

$$a_1=3300.\left(1+\frac{9}{11}.\ \phi\right) \tag{5}$$

If 3 is the factor of safety,

$$\therefore b=1100.\left(1+\frac{9}{11}.\ \phi\right) \tag{6}$$

Example.–The stresses upon a bar of Phœnix axle-iron, normal to its cross-section, vary from a maximum tension of $50{,}000^k$ to a minimum tension of $20{,}000^k$; determine the admissible stress per cent.[2] and the necessary sectional area.

$$\text{By (4), } b=700.\left(1+\frac{1}{2}.\frac{20{,}000}{50{,}000}\right)=840^k \text{ per cent.}^2$$

$$\text{and } \therefore\ F=\frac{50{,}000}{b}=\frac{50{,}000}{840}=59.52.\ \text{sq. cent.}^{res}.$$

Let p be the dead load, and q the total load, per lineal unit of length, upon the flanges of roof and bridge trusses.

$\therefore\ \phi=\frac{p}{q}$, and equations (4) and (6) respectively become,

$$b=700.\left(1+\frac{1}{2}.\frac{p}{q}\right) \tag{7}$$

$$b=1100.\left(1+\frac{9}{11}.\frac{p}{q}\right) \tag{8}$$

Example.–Determine the limiting stress per cent.[2] for the flanges of a wrought iron lattice girder when the ratio of the dead load to the greatest total load is $\frac{1}{3\frac{1}{2}}$.

$$\text{By (7), } b=700\left(1+\frac{1}{2}.\frac{1}{3\frac{1}{2}}\right)=800^k.$$

(12).—*Weyrauch's Formula.*—Let a bar of a unit sectional area be subjected to stresses which are alternately different in kind, and which vary from an absolute numerical maximum a' (= max. B) of the one kind to a maximum a'' (= max. B[1]) of the other kind.

Let $a' + a'' = d$ = the maximum numerical difference of stress.

$$\text{Let } \frac{a''}{a'} = \frac{max.\ B'}{max.\ B} = \phi'$$

$$\text{If } a''=0, \therefore a' = d = u$$

$$\text{If } a''=s, \therefore a' = s = \frac{d}{2}$$

By Wöhler's Law, $a' = d = f.d$ (9)

f being an unknown co-efficient of which the value remains to be determined.

$$\text{If } a'=u, \therefore f = 1$$

$$\text{If } a'=s, \therefore f = \frac{1}{2}$$

Weyrauch's assumption, viz., $f = \dfrac{u-s}{2.u-s-a'}$, satisfies these extreme conditions, the most reliable results of the few experiments yet recorded, and also Wöhler's deduction that a' diminishes as d increases and *vice versa.*

Hence (9) becomes,

$$a' = \frac{u-s}{2\ u-s-a'}.\ d = \frac{u-s}{2\ u-s-a'}.\ (a'+a'')$$

$$\text{and } \therefore a' = u.\left(1 - \frac{u-s}{u}.\frac{a''}{a'}\right) = u.\left(1 - \frac{u-s}{u}.\ \phi'\right) \qquad (10)$$

This is Weyrauch's formula, and may be always applied in those cases in which a member is subjected to stresses alternating between tension and compression, or between shearing actions in opposite directions.

In the Phœnix iron experiments already referred to, it was found that $s = 1170^k$ per cent.[2], $\therefore \dfrac{u-s}{u} = \dfrac{7}{15}$

Taking $u = 2100^k$ as before, and making $\dfrac{u-s}{u} = \dfrac{1}{2}$, (10) becomes,

$$a' = 2100.\left(1 - \frac{\phi'}{2}\right) \qquad (11)$$

If 3 is the factor of safety,

$$\therefore b = 700.\left(1 - \frac{\phi'}{2}\right) \qquad (12)$$

In the steel experiments, Wöhler found that $s=2050^k$ per cent.[2]

$$\therefore \frac{u-s}{u}=\frac{5}{12}$$

Taking $u=3300^k$, and $s=1800^k$, $\therefore \frac{u-s}{u}=\frac{5}{11}$, and (10) becomes

$$a'=3300.\left(1-\frac{5}{11}.\varphi'\right) \quad (13)$$

If (3) is the factor of safety,

$$\therefore b=1100.\left(1-\frac{5}{11}.\varphi'\right) \quad (14)$$

If a very soft steel be employed in the construction of a bridge, it may be advisable to diminish still further the admissible stress per unit of sectional area. For example, it may be assumed that $t=5200^k$, $u=3000^k$, and $s=1500^k$, so that (2) and (10) respectively become,

$$a_1=3000.\left(1+\frac{3}{4}.\varphi\right) \quad (15)$$

$$\text{and } a'=3000.\left(1-\frac{1}{2}.\varphi'\right) \quad (16)$$

Example.—The stresses upon a wrought iron bar normal to its cross-section vary between a tension of $40{,}000^k$ and a compression of $30{,}000^k$; find the sectional area (disregarding buckling).

$$\text{By (12), } b=700.\left(1-\frac{1}{2}.\frac{30{,}000}{40{,}000}\right)=437.5^k \text{ per cent.}^2$$

$$\therefore F=\frac{40{,}000}{437.5}=91.42 \text{ sq. cent}^{res}.$$

Shearing Stresses.—For shearing stresses in opposite directions, Wöhler found, in the case of Krupp cast steel (untempered), that $u=2780^k$ per cent.[2] and $s=1610^k$ per cent.[2], *i.e.*, about $\frac{4}{5}$ of the corresponding values for stresses which are alternately tensile and compressive, and it may be generally assumed that the value of b for shearing stresses is $\frac{4}{5}$ of its value for stresses which are alternately tensile and compressive and which have the same ratio φ'.

(13).—*Remarks upon the values of t, u, s and b.*—As yet the value of u in compression has not been satisfactorily determined, and for the present its value may be assumed to be the same both in tension and compression.

If, as Wöhler states, "repeated stresses" are detrimental to the strength of a material, then the values of u and s diminish as the repetitions in-

crease in number, and are minima in structures designed for a practically unlimited life.

Only a very few of Wöhler's experiments give the values of t, u, s, and a, so that Launhardt's and Weyrauch's assumptions for the value of f must be regarded as tentative only, and require to be verified by further experiments. The close agreement of Wöhler's results from tests upon untempered cast-steel (Krupp) with those given by Launhardt's formula may be seen from the following:—

For $t=1100$ centners* per sq. zoll, Wöhler found that $u=500$ centners per sq zoll. Thus (2) becomes,

$$a_1 = 500.\left(1 + \frac{6}{5}.\frac{a_2}{a_1}\right)$$

$$\text{and } \therefore\ a_1^2 - 500.a_1 - 600.a_2 = 0$$

Hence, for $a_2 = 0$, 250, 400, 600, 1100

Launhardt's formula gives $a_1 = 500$, 710, 800, 900, 1100

Wohler's experiments gave $a_1 = 500$, 700, 800, 900, 1100

Again, with Phœnix iron for $t=500$ centners per sq. zoll, u was found to be 300 centners per sq. zoll.

$$\text{and } \therefore\ a_1 = 300\left(1 + \frac{5}{6}.\frac{a_2}{a_1}\right)$$

$$\text{or } a_1^2 - 300\ a_1 - 250.\ a_2 = 0$$

If $a_2 = 240$, $\therefore\ a_1 = 436.8$, which almost exactly agrees with the result given by the tension experiments.

In general, the admissible stress per sq. unit of sectional area may be expressed in the form,

$$b = v.(1 \pm m.\ \phi) \tag{17}$$

v and m being certain co-efficients which depend upon the nature of the material and also upon the manner of the loading. Consider three cases, the material in each case being wrought-iron.

(a).–Let the stresses vary between a maximum compression and an equal maximum compression; $\therefore\ \phi=1$

$$\text{and } \therefore\ b = 700.\left(1 - \frac{1}{2}\right) = 50^k \text{ per cent.}^2$$

(b).–Let the material be subjected to stresses which are either tensile or compressive, and let it always return to the original unstrained condition.

$$\therefore \text{ min. } B = 0, \text{ or max. } B' = 0 \text{ and } \therefore\ \varphi = 0.$$

$$\therefore\ b = 700\ (1 \pm 0) = 700^k \text{ per cent.}^2$$

* A centner = 110.23 pounds. A square zoll = 1.0603 sq. ins.

(c).—Let the material be continually subjected to the same dead load, $\therefore$ min.B=max.B,

$$\text{and } \therefore \; b=700.(1+\tfrac{1}{2})=1050^k \text{ per cent.}^2$$

Thus, in these three cases, the admissible stresses are in the ratio of 1:2:3, a ratio which has been already adopted in machine construction as the result of experience.

Wöhler, from his experiments upon untempered cast-steel (Krupp), concluded that for alternations between an unloaded condition and either a tension or a compression $b=1100$, and for alternations between equal compressive and tensile stresses $b=580$.

In America, it has often been the practice to take

$$F=\frac{max.\ B + max.\ B'}{700}=\frac{a'+a''}{700}$$

for stresses alternately tensile and compressive, it being assumed that if the stresses are tensile only their admissible values may vary from 0^k to 700^k per cent.[2]

$$\text{Since } \phi'=\frac{a''}{a'}, \; \therefore \; a'=\frac{700.F}{1+\phi'}$$

$$\text{and } \therefore \; b=\frac{a'}{F}=\frac{700}{1+\phi'},$$

Comparing this with (12),

for $\phi'=0$ $\quad \frac{1}{4}$ $\quad \frac{1}{2}$ $\quad \frac{3}{4}$ $\quad 1$,

(18) gives $b=700$, 560, 467, 400, 350,

and (12) gives $b=700$, 612, 525, 437, 350.

Note.—For further information on the subject treated of in articles 12 to 15, the reader is referred to Weyrauch's Fertigkeit und Dimensionen-berechnung (translated by Prof. A. Jay DuBois), from which they have been substantially taken.

Table of the Strengths, Elasticities, Co-efficients of Rupture (C), and Weights of different Materials.

MATERIALS.	Tensile strength in tons per sq. in.	Compressive strength in tons per sq. in.	Shearing strength in tons per sq. in.	Ductility per cent. of original length.	Elastic limit per cent. of tensile strength.	Co-efficient of Direct Elasticity in lbs. per sq. in.	Co-efficient of transverse Elasticity in lbs. per sq. in.	Values of C in lbs. for a load at centre.	Weight in lbs. per cubic ft.
Cast Iron.									
Various qualities........	6 to 13	35 to 65	8 to 12.37 (rarely 18)	....	....	14,000,000 to 29,000,000			
Average	$7\frac{1}{3}$	50	9	....	about $33\frac{1}{3}$	17,000,000	2,850,000	30,000 to 43,500	450
Average market value....	6	38	9						
Very good	11 to $13\frac{1}{2}$			1					
Wrought Iron.									
Low average	....	16	20						
Various qualities of bar..	20 to 26	13 to 33	$14\frac{1}{2}$ to 21	....	....	29,000,000	8,500,000 to 10,000,000	33,000 to 58,000	480
Market Bars, full size....	$20\frac{1}{4}$ to $22\frac{1}{3}$	....	....	$12\frac{1}{2}$	about 50				
" " prepared specimens...........	$22\frac{1}{3}$ to $23\frac{1}{5}$								
L*, T*, and other sections.	$22\frac{1}{3}$	....	....	15	" 50				
Various qualities of plate.	$20\frac{1}{2}$ to $22\frac{1}{3}$	....	....	15	" 60				
" prepared specimens.	$21\frac{1}{2}$								
Plates punched...........	....	....	$22\frac{1}{2}$ to 27						
Hammered Scrap punch'd	....	....	20 to 23						
*Iron Wire..............	31 to $44\frac{1}{3}$	....	....	....	....	25,300,000			
Wire Rope..............	40	....	....	....	....	15,000,000			
Double Riveted Joints....	15.9								
Single " " ...	12.7								

*Mr. H. Tomlinson has proved that the co-efficient of elasticity of an Iron Wire which has suffered permanent extension, may be considerably increased by allowing it to rest for some hours, either loaded or unloaded, and that the increase is not attended with any appreciable increase of density.

Table of the Strengths, Elasticities, Co-efficients of Rupture (C), and Weights of different Materials.—*Continued.*

MATERIALS.	Tensile strength in tons per sq. in.	Compressive strength in tons per sq. in.	Shearing strength in tons per sq. in.	Ductility per cent. of original length.	Elastic limit per cent. of Tensile strength.	Co-efficient of Direct Elasticity in lbs. per sq. in.	Co-efficient of transverse Elasticity in lbs. per sq. in.	Values of C in lbs. for a load at centre.	Weight in lbs. per cubic ft.
Steel.									
Bars	29 to 59	....	....	4.7 to 36	....	29,000,000 to 42,000,000			
Piston Rod Steel.........	33.7	....	....	.9	79				
Plates, L's and T's for ordinary structural purposes	30 to 40	30 to 40	....	4.4 to 38	50				
Do special purposes...	40 to 55								
Landore Plates..........	30	....	....	....	43				
Mild Steel Plates........	26 to 30	....	....	20	57.7 to 90				
" " Average.......	28.16	....	....	24.25	74				
Average Bessemer Steels.	33.43	....	....	13.4	59.8	30,000,000	8,534,000 to 14,223,000	80,000 to 129,000	489.6 or 490
" Crucible Steels..	36.30			7.85	55				
Low average for Soft Unhardened Cast Steel	32	30	24						
Unhardened Steel.........	....	89							
Hardened Steel (low temper)............	....	158							
Hardened Steel (high temper)............	....	166							
Compressed Steel, according to requirement	40 to 72	....	....	32 to 14					
Chisel Steel, according to temper.	56 to 96	....	....	10 to 3.3					
Chrome Steel............									
Phosphor Bronze.	26	....	....	....	....	14,000,000			

Phosphor Bronze.	26	....	....	....	....	14,000,000				

Table of the Strengths, Elasticities, Co-efficients of Rupture (C), and Weights of different Materials.—*Continued.*

MATERIALS.	Tensile strength in tons per sq. in.		Compressive strength in tons per sq. in. Along grain.		Force in tons per sq. in. to crush fibres transvers'ly 1-20 in. deep.	Shearing strength in tons per sq. in.		Co-effic'nt of Direct Elasticity in lbs. per sq. in.	Co-effic'nt of transverse Elasticity in lbs. p. sq. in	Values of C in lbs. for load at centre.	Weight in lbs. per cubic ft.
	Al'ng gr'n	Ac'ss gr'n	Dry.	Very dry.		Al. grn.	Ac'ss gr'n				
Timber.											
Alder	4.5 to 6.3	....	3.05	3.17	....	....	....	....	....	....	50
Apple	8.8	....	....	....	....	....	....	....	....	5,300 to 7,000	50
Ash, English	1.8 to 7.6	....	3.87	4.16	....	....	.625	1,525,000 to 2,290,000	76,000	8,000	43 to 53
Ash, Canadian	2.45	....	....	2. 5	1.03	....	....	1,380,000	....	7,000	47
Beech	2.1 to 9.8	....	3.45	4.16	....	....	....	1,350,000	....	4,500 to 6,000	43 to 53
Birch	3.14to6.7	....	1.45 Eng	5.24 Am. 2.85 Eng	....	....	....	1,645,000	....	5,750	45 to 49
Box	6.74	....	3.6	4.5	....	....	....	1,800,000	....	5,700	64
Cedar	1.3 to 5.1	....	2.53	2.61	....	....	....	486,000	....	5,300	35 to 47
Chestnut	3.12to5.8	....	....	2.53	.42	.308	....	1,140,000	....	4,600	35 to 41
Elm, English	2.4 to 6.3	....	....	4.6	....	....	.625	700,000 to 1,340,000	76,000	2,400 to 4,850	34 to 37
Elm, Canadian	4.1	....	....	....	....	....	....	2,470,000	....	7,250	47
Fir. New England	2.23to4.5	....	....	....	....	....	....	1,200,000	....	4,600	
Fir. Norway	....	....	....	....	....	....	....	1,800,000	....	4,600	32
Fir. Riga	1.8 to 5.5	....	....	....	....	....	....	870,000 to 3,000,000	....	4,600	34 to 47
Fir. Spruce (Am.)	13) 4.5	....	....	....	.22	.21	.27	1,200,000 to 1,800,000	....	4,000 to 6,000	29 to 32

Table of the Strengths, Elasticities, Co-efficients of Rupture (C), and Weights of different Materials.—*Continued.*

MATERIALS.	Tensile strength in tons per sq. in.		Compressive strength in tons per sq. in.		Force in tons per sq. in. to crush fibres transversally 1-20 in. deep.	Shearing strength in tons per sq. in.		Co-effic'nt of Direct Elasticity in lbs. per sq. in.	Co-effic'nt of transverse Elasticity in lbs. p. sq. in	Value of C in lbs. for a load at centre.	Weight in lbs. per cubic ft.
	Al'ng gr'n	Ac'ss gr'n	Along grain. Dry.	Very dry.		Al. grn.	Ac'ss gr'n				
Timber—*Continued.*											
Greenheart	2.7 to 4.1	....	4.46to6.5	....	....	......	....	1,700,000	....	6,660 to 13,750	58 to 72
Hemlock	....	....	....	....	....	.24					
Hornbeam.........	9.1	....	2	3.25	....	......	....	....	....	2,834	47.5
Lancewood	3.6 to 6.7	....	....	....	....	......	....	....	....	10,000	42 to 63
Larch............	1.9 to 5.3	.43 to .76	1.43	2.48	....	......	.43 to .76	1,360,000	....	5,300 Eng. 7,000 Am.	32 to 38
Lignum Vitæ	4.46to5.3	....	2.57	4.28	....	......	....	1,000,000	....	8,000	41 to 83
Locust	4.5 to 6.7	....	1.33	2.45	....	.535	....	....	....	8,000	
Mahogany, Spanish	1·7 to 7.3	....	3.3	3.3	....	......	....	1,255,000 to 3,000,000	....	3,800 to 5,300	53
Mahogany, Hondur.	1.3 to 3.6	....	....	....	....	......	....	1,596,000 to 1,970,000	....	6,600	35
Maple............	7.7	....	2.23	2.68	....	......	....	....	....	....	49
Mora	4.1	....	4.4	....	....	......	....	1,860,000	....	10,500 to 11,000	57 to 68
Oak, English	3.4 to 5.4	1.03	2.89	4.49	.. ..	3.48	1.34 to 2.23	1,200,000 to 1,750,000	82,000	6,700	49 to 58
Oak, American....	3 to 4.6	....	1.89	2.68	....	......	....	2,100,000	68,912	6,700	61

Table of the Strengths, Elasticities, Co-efficients of Rupture (C), and Weights of different Materials.—*Continued.*

MATERIALS.	Tensile strength in tons per sq. in.		Compressive strength in tons per sq. in.		Force in tons per sq. in. to crush fibres transversally 1-20 in. deep.	Shearing strength in tons per sq. in.		Co-effic'nt of Direct Elasticity in lbs. per sq. in.	Co-effic'nt of transverse Elasticity in lbs p.sq.in	Value of C in lbs. for a load at centre.	Weight in lbs. per cubic ft.
	Al'ng gr'n	Ac'ss gr'n	Along grain. Dry	Along grain. Very dry.		Al. grn.	Ac'ss gr'n				
Timber—*Continued.*											
Pine											
Pitch (Am.)..	2.1 to 4.4	....	3	3	....	......	....	1,252,000 to 3,000,000	....	4,590 to 7,000	41 to 58
Red (Am.)....	1.7 to 3.6	....	2.4to2.77	3.35	....	......	.22 to .36	1,960,000 to 2,350,000	62,000 to 116,000	5,700	34
Yellow(Am.)..	2.2 to 5.3	....	2.4	2.43	.27	.227	....	1,600,000 to 2,480,000	....	4,700	32
White (Am.)..	1.3 to 3.3										
Dantzic	1.4 to 4.5	....	....	....	....	......	.22	2,300,000	....	6,150	36
Kaurie	2	.25	....	....	....	......	....	2,800,000	....	4,435 to 5,680	34
Memel........	4.2 to 4.9	.24to.375	....	....	....	......	....	....	....	....	34
Plane	5.4	....	....	....	....	......	....	1,343,250	....	....	40
Poplar	2.68to3.2	.79	1.38	2.28	....	......	....	763,000	....	....	23 to 26
Sycamore	4.3 to 5.8 3.6 (old)	....	3.16	....	....	......	....	1,040,000	....	4,800	36 to 43
Teak	4.7 to 6.7	....	5.35	5.4	....	......	....	2,100,000 to 2,414,000	....	6,000 to 9,500	41 to 52
Walnut...........	3. 5	....	2.7	3.2	....	......	....	....	....	8,000	38 to 57
Willow	4.5to6.25	....	1.3	2.7	....	......	....	1,400,000	....	3,300 to 4,700	24 to 35

Table of Factors of Safety of Cast Iron, Wrought Iron, Steel, and Timber.

MATERIALS.	USE.	Nature of Load.	Value of Factor of Safety.
Cast Iron	In Girders	Dead.	3 to 6
	" Water Tanks	Do	4
	" Pillars	Do	6
	" Pillars and Machinery	Live.	8 to 10
	" " liable to transverse shock	Do	10
Wrought Iron	In Girders	Dead.	3
	" Compression Bars	Do	4
	" Girders	Live.	6
	" Compression Bars	Do	6 to 8
	" Bridge Tension Bars	Mixed.	4 or 5
	" Bridge Compression Bars	Do	6 to 8
	" Roofs	Do	4 to 7
Steel	In Bridges	Mixed.	3 to 5
Timber	In various structures	Dead, Live or Mixed.	4 to 14 av. 10

NOTE.—The safe *working tensile* load of the white and yellow pines, of spruce, oak and such woods as are employed in the construction of bridges, roofs, &c., may be assumed to vary, according to the character of the structure, from 1,000 to 2,000-lbs. per sq. in.

The safe *working compressive* load diminishes as the length of the timber member increases, and may be assumed to vary from 1,000 to 150-lbs. per sq. in.

The safe *working shearing* load, along the grain, is about 100-lbs. per sq. in. for oak, -lbs. per sq. in. for beech, and 55-lbs. per sq. in. for pine.

TABLE III.

Table of the Linear Expansion of Solids per unit of length by Heat.

MATERIALS.	From 32° F. to 212° F.	From 32° F. to 572° F.
Glass	.00077 to .00101	
Baywood (along the grain)	.00046 to .00057	
Deal (along the grain)	.00043 to .00044	
Cast Iron	.00111	
Wrought Iron	.00125	.00145
Steel	.00108 to .00125	
Tin	.0022	
Zinc	.0025 to .00294	
Brass	.002	
Copper	.00178	.00188
Bronze	.00181	
Gun-Metal	.00181 to .00191	
Platinum	.00087	.00092
Silver	.00195	
Gold	.00152 to .00155	

EXAMPLES.

(1).–An iron bar of uniform section and 10-ft. in length, stretches .012-in. under a unit stress of 25,000-lbs.; find E.

(2).–A ship at the end of a 600-ft. cable and one at the end of a 500-ft. cable stretch the cables 3-ins. and $2\frac{1}{2}$-ins. respectively; what are the corresponding strains ?

(3).–A rectangular timber tie is 12-ins. deep and 40-ft. long. If $E = 1,200,000$-lbs., find the proper thickness of the tie so that its elongation under a pull of 270,000-lbs. may not exceed 1.2-ins.

(4).–A roof tie-rod 142-ft. in length and 4-sq. ins. in sectional area is subjected to a stress of 80,000-lbs. If $E = 30,000,000$-lbs., find the elongation of the rod and the corresponding work.

(5).–An iron wire 1-8-in. in diameter and 250-ft. in length is subjected to a tension of 600-lbs., the consequent strain being $\frac{1}{300}$; find E, and shew by a diagram the amount of work done in stretching the wire within the limits of elasticity.

(6).–A steel rod 100-ft. in length has to bear a weight of 4000-lbs. If $E = 35,000,000$-lbs., and if the safe strain is .0005, determine the sectional area of the rod, and the work of extension, (1).–When the weight of the rod is neglected, (2).–When the weight of the rod is taken into account.

(7).–Find the work done in raising a Venetian blind.

(8).–The length of a cast-iron pillar is diminished from 20-ft. to 19.97-ft. under a given load; find the strain and the compressive unit stress, E being 17,000,000-lbs.

(9).–A rectangular timber strut 24-sq. ins. in sectional area and 6-ft. in length is subjected to a compression of 14,400-lbs.; determine the diminution of the length, E being 1,200,000-lbs.

(10).–A brick wall, 2-ft. thick, 12-ft. high, and weighing 112-lbs. per cubic ft., is supported upon solid pitch pine columns 9-ins. in diameter, 10-ft. in length, and spaced 12-ft. centre to centre; find the compressive unit stress in the columns, (1).–At the head (2).–At the base; the timber weighs 50-lbs. per cubic foot.

If the crushing stress of pitch pine is 5300-lbs. per sq. in., and the factor of safety 10, find the height to which the wall may be built.

(11).–Determine the diameter of the wrought iron columns that might be substituted for the timber columns in question (10),–7500-lbs. per sq. in. being the safe compressive stress of wrought iron.

(12).–A timber pillar 30-ft. in length has to support a beam at a point 30-ft. from the ground; if the greatest safe strain of the timber is $\frac{1}{300}$, what thickness of wedge should be driven between the head of the pillar and the beam?

(13).–An hydraulic hoist rod 50-ft. in length and 1-in. in diameter is attached to a plunger 4-ins. in diameter upon which the pressure is 800-lbs. per sq. in.; determine the altered length of the rod, E being 30,000,000-lbs. 50.0213

(14).–How many $\frac{7}{8}$-in. rivets must be used to join two wrought iron plates, each 36-ins. wide and $\frac{1}{2}$-in. thick, so that the rivets may be as strong as the riveted plates, the tensile and shearing strength of wrought iron being in the ratio of 10 to 9?

(15).–A wrought iron bar 25-ft. in length and 1-sq. in. in sectional area stretches .0001745-ft. for each increase of 1° F. in the temperature, determine the work done by an increase of 20° F.

How may this property of extension under heat be utilised in straightening walls that have fallen out of plumb?

(16).–A wrought iron bar 2-sq. ins. in sectional area has its ends fixed between two immovable blocks when the temperature is at 32° F.; what pressure will be exerted upon the blocks when the temperature is 100° F.?

(17).–A line of rails is 10 miles in length, when the temperature is at 32° F., determine the length when the temperature is at 100° F.

(18).–The dead load of a bridge is 5-tons and the live load 10-tons per panel, the corresponding factors of safety being 3 and 6; find the *compound* factor of safety.

(19).–The dead load upon a short hollow cast-iron pillar with a sectional area of 20-sq. ins. is 50-tons. If the strain in the metal is not to exceed .0015, find the greatest live load to which the pillar might be subjected.

(20).–A steel suspension rod 30-ft in length and $\frac{1}{2}$-sq. in. in sectional area carries 3500-lbs. of the roadway and 3000-lbs. of the live load; determine the *gross* load and also the extension of the rod; E being 35,000,000-lbs.

(21).–What form does the useful work done by a hammer take when a nail is driven in? What becomes of the rest of the energy of the mass of the hammer after striking the blow? A hammer weighing 10-lbs. strikes a blow of 10-ft. lbs. and drives a nail $\frac{1}{2}$-inch into a piece of timber; find the speed with which the hammer moves, and the mean resistance to entry. R = 240

Find the amount of a steady pressure that will produce the same effect as the hammer. 480

(22).–A steel rod 10-ft. in length and $\frac{1}{2}$-sq. in. in sectional area is strained to the proof by a tension of 25,000-lbs.; find the resilience of the rod, E being 35,000,000-lbs. 178$\frac{4}{7}$ ft-lbs

(23).–A wrought iron rod 25-ft. in length and 1-sq. in. in sectional area is subjected to a steady stress of 5000-lbs.; what amount of live load will instantaneously elongate the rod by $\frac{1}{8}$-in., E being 30,000,000-lbs? 6250

(24).–Determine the shortest length of a metal bar a-sq. ins. in sectional area that will safely resist the shock of a weight of W-lbs., falling a distance of h-ft. Apply the result to the case of a steel bar 1-sq. in. in sectional area, the weight being 50-lbs, the distance 16-ft., the proof strain, $\frac{1}{700}$, and E=35,000,000-lbs.

(25).–A signal wire 2000-ft. in length and $\frac{1}{8}$-in. in diameter is subjected to a steady stress of 300-lbs. The lever is suddenly pulled back, and the corresponding end of the wire moves through a distance of 4-ins.; determine the instantaneous increase of stress.

If the total back weight is 350-lbs., what is the range of the signal end of the wire?

(26).–A steel rod has its upper end fixed and hangs vertically. The rod is tested by means of a ring weighing 60-lbs. which slides along the rod and is checked by a collar screwed to the lower end. A scale is marked upon the rod with the zero at the fixed end. If the strain in the steel is not to exceed $\frac{1}{700}$, what is the reading from which the weight is to be dropped? What should be the reading of the collar?

(27).–Steam at a pressure of 50-lbs. per sq. in. is suddenly admitted upon a piston 32-ins. in diameter. The steel piston rod is 48-ins. in length and 2-ins. in diameter, E being 35,000,000-lbs.; find the work done upon the rod.

What should be the pressure of admission to strain the rod to a proof of .001?

(28).–A pitch pine pile 14-ins. square is 20-ft. above ground, and is being driven by a falling weight of 112-lbs.; find the fall so that the inch stress at the head of the pile may be less than 800-lbs. 746 nearly

Supposing that the pile sinks 2-ins. into the ground, by how much would it be safe to increase the fall? 118.5

(29).–A shock of N-ft. lbs. is safely borne by a bar l-ft. in length and a-sq. ins. in sectional area; determine the increased shock which the bar will bear when the sectional area of the last mth of its length is increased to $r.a$.

(30).–A boulder grappler is raised and lowered by a wire rope 1-in. in diameter hanging in double sheaves. On one occasion a length of 150-ft. of rope was in operation, the distance from the winch to the upper block being 30-ft. The grappler laid hold of a boulder weighing 10-tons, what was the extension of the rope, E being 15,000,000-lbs.?

The boulder suddenly slipped and fell a distance of 6-ins. before it was again held, find the maximum stress upon the rope.

What weight of boulder may be lifted, if the proof stress in the rope is not to exceed 25,000-lbs per sq. in. of *gross* sectional area?

(31).–A chain l-ft. in length and a-sq. ins. in sectional area has one end securely anchored, and suddenly checks a weight of W-lbs. attached to the other end, and moving with a velocity of V-ft. per second away from the anchorage; find the greatest pull upon the chain.

Apply the result to the case of a wagon weighing 4-tons, and worked from a stationary engine by a rope 3-sq. ins. in sectional area. The wagon is running down an incline at the rate of 4 miles an hour, and, after 600-ft. of rope have been paid out, is suddenly checked by the stoppage or reversal of the engines, ($E = 15,000,000$ lbs.).

(32).–A chain l-ft. in length and a-sq. ins. in sectional area has one end attached to a weight of W-lbs. at rest, and at the other end is a weight of $n.W$-lbs. moving with a velocity of V-ft. per second and away from the first; find the greatest pull on the chain.

(33).–A dead weight of 10-tons is to act as a drag upon a ship to which it is attached by a wire rope 150-ft. in length, and having an effective sectional area of 8-sq. ins. If the velocity of the floating ship is 20-ft. per second, and if its inertia is equivalent to a mass of 390-tons, find the greatest pull on the chain. (E=15,000,000-lbs.)

(34).–The steady thrust or pull upon a prismatic bar is suddenly reversed, shew that its effect is trebled.

(35).–In a circular pipe of internal radius r and thickness t, a column of water of length l, flowing with a velocity due to the head h, is *suddenly* checked; shew that,

$$g.\,h = \frac{E.\,t.\,\lambda^2}{r}.\left\{1 + \frac{1}{2}.\frac{t}{r}.\left(1 + \frac{E}{E_1}\right) + \frac{t^2}{r^2}\right\}$$

E being the coeff. of elasticity of the pipe material, E_1 the coefficient of compressibility of the water, and λ the extension of the pipe circumference due to E.

CHAPTER A.

(1).—*On the extension of a prismatic bar.*—The elementary law of extension is sometimes enunciated as follows:—

A prismatic bar of length L, and sectional area A is stretched, and its length is $L+x$ when the force of extension is P; if dP is the increment of force corresponding to an increment dx of length,

$$\therefore dP = E.A.\frac{dx}{L+x}$$

Hence, the force producing an extension l is equal to

$$\int_0^l E.A.\frac{dx}{L+x} = E.A.\log_e\left(1+\frac{l}{L}\right) = P_1, \text{ suppose.}$$

But $\log_e\left(1+\frac{l}{L}\right) = \frac{l}{L} - \frac{1}{2}.\left(\frac{l}{L}\right)^2 + \frac{1}{3}.\left(\frac{l}{L}\right)^3 - \ldots = \frac{l}{L}$, approximately.

$$\therefore P_1 = E.A.\frac{l}{L}$$

Corollary.—From the last equation, $\frac{dP_1}{dl} = \frac{E.A}{L}$, and $\frac{E.A}{L}$ is consequently a measure of the longitudinal *stiffness* of a bar, so that for the *same* material, the stiffness varies directly as the sectional area and inversely as the length, while for *different* materials, it also varies directly as the co-efficient of elasticity.

(2).—*Work of Extension.*—The force producing the increment dx has for its least value $P\left(=E.A.\frac{x}{L}\right)$, for its greatest value $P+dP$, and for its mean value $P+\frac{dP}{2}$, so that the *work done* is $\left(P+\frac{dP}{2}\right).dx = P.dx$, approximately.

Hence, the *work done* in stretching the bar until its length is $L+l$ is equal to

$$\int_0^l P.dx = \int_0^l E.A.\frac{x}{L}.dx = \frac{E.A}{L}.\frac{l^2}{2}.$$

(3).–*On the oscillatory motion of a weight at the end of a vertical elastic rod.*–An elastic rod of natural length L (OA) and sectional area A, is suspended from O, and carries a weight P at its lower end, which elongates the rod until its length is $OB = L + l$.

Assume that the mass of the rod, as compared with P, is sufficiently small to be disregarded,

$$\therefore P = E.\,A.\frac{l}{L}.$$

If the weight is made to descend to a point C, and is then left free to return to its state of equilibrium, it must necessarily describe a series of vertical oscillations about B as centre.

Take B as the origin, and at any time t let the weight be at M distant x from B; also let $BC = c$.

Two cases may be considered.

First, suppose the end of the rod to be *gradually* forced down to C and then suddenly released.

According to the principle of *vis viva*,

$$\frac{P}{g}.\frac{1}{2}.\left(\frac{dx}{dt}\right)^2 = \text{the } \textit{work done} \text{ between } C \text{ and } M = \frac{E.\,A}{L}.\left(\frac{c^2}{2} - \frac{x^2}{2}\right)$$

$$\text{or, } \frac{P}{g}.\frac{1}{2}.\left(\frac{dx}{dt}\right)^2 = \frac{P}{l}.\frac{1}{2}.\,(c^2 - x^2)$$

$$\text{and } \therefore v, \text{ the velocity of the weight at } P = \sqrt{\frac{g}{l}}.(c^2 - x^2)^{\frac{1}{2}}$$

Now v is zero when $x = \pm c$, so that the weight will rise above B to a point C_1 where $B\,C_1 = c = BC$.

Again, from the last equation, $dt.\sqrt{\frac{g}{l}} = \frac{dx}{(c^2 - x^2)^{\frac{1}{2}}}$, and integrating between the limits 0 and x, $\therefore\ t = \sqrt{\frac{g}{l}} = sin^{-1}.\frac{x}{c}$, and the oscillations are therefore isochronous.

When $x = c$, $t = \frac{\pi}{2}.\sqrt{\frac{l}{g}}$, and the time of a complete oscillation is $\pi.\sqrt{\frac{l}{g}}$

Next, suppose the oscillatory motion to be caused by a weight P falling without friction from a point D, and being suddenly checked and held by a catch at the lower end of the rod.

Take the same origin and data as before, and let $AD=h$.

The elastic resistance of the rod at the time t is $E.A.\frac{l+x}{L}$, and the equation of motion of the weight is,

$$\frac{P}{g}\cdot\frac{d^2x}{dt^2}=P-E.A.\frac{l+x}{L}=P-\frac{P}{l}\cdot(l+x)=-P.\frac{x}{l}$$

$$\text{or } \frac{d^2x}{dt^2}=-\frac{g}{l}.x$$

Integrating, $\therefore\left(\frac{dx}{dt}\right)^2=-\frac{g}{l}\cdot x^2+c_1$, c_1 being a constant of integration.

But $\frac{dx}{dt}$ is zero, when $x=c$, and $\therefore c_1=\frac{g}{l}\cdot c^2$

$$\text{Hence, } \left(\frac{dx}{dt}\right)^2=\frac{g}{l}\cdot(c^2-x^2).=v^2$$

This is precisely the same equation as was obtained in the first case, and between the limits 0 and x, $t.\sqrt{\frac{g}{l}}=sin^{-1}.\frac{x}{c}$, so that the motion is isochronous, and the time of a complete oscillation is $\pi.\sqrt{\frac{l}{g}}$

Cor. 1.–When $x=-l$, $\left(\frac{dx}{dt}\right)^2=2.g.h$, and $\therefore\frac{g}{l}\cdot(c^2-l^2)=2.g.h.$, or $c^2=l^2+2.l.h.$

Cor. 2.–If $h=0$, *i.e.*, if the weight is merely placed upon the rod at the end A, $\therefore c=\pm l$, and the amplitude of the oscillation is *twice* the statical elongation due to P.

Cor. 3.–The rod may be safely stretched until its length is $L+l$, while a further elongation c might prove most injurious to its elasticity, which shews the detrimental effect of vibratory motion. If the weight is applied at irregular intervals, the amplitude of the oscillations will be increased, until at last rupture would take place. The failure of the Broughton Suspension Bridge is said to have been due to this cause.

Cor. 4.–The co-efficient of elasticity of the rod may be approximately found by means of the formula $T=\pi.\sqrt{\frac{l}{g}}$, T being the time of a complete oscillation. For, suppose that the rod emits a musical note of n vibrations per second, $\therefore\ \pi.\sqrt{\frac{l}{g}}=T=\frac{1}{2.n}$, is the time of travel from C to C_1,

$$\therefore l=\frac{g}{4\pi^2.n^2}, \text{ and hence, } E=\frac{P.L}{A}\cdot\frac{4.\pi^2.n^2}{g}$$

Cor. 5.–Suppose that the weight is perfectly free to slide along the rod. When it returns to A, it will leave the end of the rod and rise with a certain initial velocity. This velocity is evidently $\sqrt{2g.h}$, and the weight accordingly ascends to D, then falls again, repeats the former operation, and so on. The equations of motion are in this case only true for values of x between $x = +c$, and $x = -l$.

4.–*On the oscillatory motion of a weight at the end of a vertical elastic rod of appreciable mass.*–Suppose the mass of the rod to be taken into account, and assume :–

(*a*).–That all the particles of the rod move in directions parallel to the axis of the rod.

(*b*).–That all the particles which at any instant are in a plane perpendicular to the axis, remain in that plane at all times.

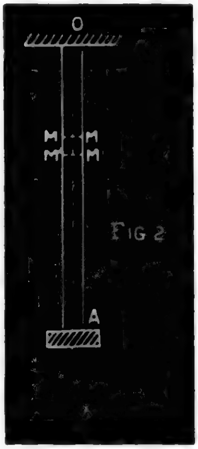


FIG 2

As before, the rod OA of natural length L and sectional A is fixed at O and carries a weight P at A

Take O as the origin, and let OX be the axis of the rod.

Let ξ, $\xi + d\xi$, and x, $x + dx$, be respectively the *actual* and *natural* distances from O of the two consecutive sections MM, $M'M'$.

Let ρ_0 be the natural density of the rod, and ρ the density of the section MM, distant ξ from O.

The forces which act upon the rod are :–

(*a*).–The upward and constant force P_0 at O.

(*b*).–The weight P_1 at A.

(*c*).–The weight of the rod.

(*d*).–A force X per unit of mass through the slice bounded by the planes MM, $M'M'$, distant and $\xi + d\xi$ respectively from O.

Suppose the rod, after equilibrium has been established, to be cut at the plane $M'M'$. In order to maintain the equilibrium of the portion $OM'M'$, it will be necessary to apply to the surface of this plane a certain force P, and the equation of equilibrium becomes,

$$-P_0 + \int_0^{\xi} \rho.A.\,d\xi.\,X + P + \rho_0.g.A\,x = 0$$

But if the thickness $d\xi$ of the slice MM' is indefinitely diminished, P is evidently the elastic reaction, and its value is

$$E.A.\frac{d\xi - dx}{dx} = E.A.\left(\frac{d\xi}{dx} - 1\right).$$

Hence, $-P_0+\int_0^{\xi}\rho.A.X.d\xi+E.A.\left(\frac{d\xi}{dx}-1\right)+\rho_0.g.A.x=0.$

Differentiating with respect to x,

$$\therefore \rho.A.X.\frac{d\xi}{dx}+E.A.\frac{d^2\xi}{dx^2}+\rho_0.g.A=0$$

But $\rho.d\xi=\rho_0.dx$

$$\therefore \rho_0.A.X.+E.A.\frac{d^2\xi}{dx^2}+\rho_0.g.A=0$$

$$\text{or } X+\frac{E}{\rho_0}.\frac{d^2\xi}{dx^2}+g=0$$

Also $\rho_0.A.X.dx$ is the resistance to acceleration arising from the inertia of the slice, and is therefore equal to $-\rho_0.A.dx.\frac{d^2\xi}{dt^2}$,

$$\text{so that } X=-\frac{d^2\xi}{dt^2}$$

$$\text{Hence, } \frac{d^2\xi}{dt^2}=\frac{E}{\rho_0}.\frac{d^2\xi}{dx^2}+g \qquad (1)$$

To solve this equation. In the state of equilibrium, $E.A.\left(\frac{d\xi}{dx}-1\right)$ is the tension in the section of which the distance from O is x, and counter-balances the weight P_1 and the weight $\rho_0.A.(l-x).g$ of the portion AMN of the rod.

$$\therefore E.A.\left(\frac{d\xi}{dx}-1\right)=P_1+\rho_0.A.g.(l-x)$$

$$\text{or } \frac{d\xi}{dx}=1+\frac{P_1}{E.A.}+\frac{\rho_0.g}{E}.(l-x)$$

$$\text{Integrating, } \therefore \xi=x+\frac{P_1}{E.A}.x+\frac{\rho_0.g}{E}.\left(l.x-\frac{x^2}{2}\right) \qquad (2)$$

There is no constant of integration as x and ξ vanish together.

This value of ξ is a *particular* solution of (1), and is independent of t.

$$\text{Put } \xi=x+\frac{P}{E.A}.x+\frac{\rho_0.g}{E}.\left(l.x-\frac{x^2}{2}\right)+z$$

z being a new function of x and t.

$$\therefore \frac{d^2\xi}{dx^2}=-\frac{\rho_0}{E}.g+\frac{d^2z}{dx^2}, \text{ and } \frac{d^2\xi}{dt^2}=\frac{d^2z}{dt^2}$$

Hence, from equation (1), $\frac{d^2z}{dt^2}=\frac{E}{\rho_0}.\frac{d^2z}{dx^2}=v_1^2.\frac{d^2z}{dx^2}$, where $v_1^2=\frac{E}{\rho_0}$

The integral of this equation is of the form,

$$z = F.(x + v_1.t) + f.(x - v_1.t),$$

or $v_1 \sqrt{\frac{E}{\rho_0}}$ being the velocity of propagation of the vibrations.

The full solution of (1) is therefore of the form,

$$\xi = x + \frac{P_1}{E.A}.x + \frac{\rho_0.g}{E}\left(l.x - \frac{x^2}{2}\right) + F.(x + v_1.t) + f.(x - v_1.t).$$

(5).—*On the flow of solids.*—The theory of the longitudinal vibrations of elastic bars shews that the amplitudes of the vibrations are twice the permanent set. This, however, is on the assumption that the mass of the bar is sufficiently small to be disregarded. Practically, the result is very different, for the amplitudes of longitudinal, and especially of transverse, vibrations rapidly diminish.

If a bar be strained even almost to rupture, and then relieved from stress and allowed to rest, it will recover a portion of its set, and, in its new state, will be capable of receiving a secondary though much smaller set.

If the bar is *steadily* strained, it may offer great resistance to elongation at first, but subsequently, the same deformation may be produced by a stress of less intensity.

Continual vibration probably increases the resistance to elongation. Thus, *time* is an important element in determining the final adjustment of a material to its permanent condition. Hence it is, that a considerable period generally elapses before suspension cables have acquired their constant set.

A consideration of these phenomena has led to the conclusion that they are due to some cause operating *within* the material and wholly independent of its elasticity. The cause has been termed the *viscosity* of the body, and is the resistance of its particles to relative motion. But this viscosity is very different from the *true viscosity* of a solid, the character of which has been most carefully investigated by M. Tresca. It has been shewn that, if the pressure upon a solid body is continually increased, the limit of elasticity is soon passed, the body becomes imperfectly elastic, and at last, under a pressure called the *fluid pressure*, the elasticity entirely disappears. The body is then said to be in a *fluid state*, and behaves precisely as a fluid, *flowing* through orifices, shewing a *contracted vein*, etc., and in consequence of the flow, retaining its specific weight unchanged.

The *general principle* of the flow of solids may be enunciated as follows:—*A pressure upon a solid body creates a tendency to the relative motion of the particles in the direction of least resistance.*

This gives an explanation of the various effects produced in materials by the operations of wire-drawing, punching, shearing, etc., and probably of the anomalous behaviour of solids under certain extreme conditions.

For example, in punching a piece of wrought iron or steel, the metal is at first compressed and *flows inwards*, while the *shearing* only commences when the opposite surface begins to open. A case brought under the notice of the author may be mentioned in illustration of this. The thickness of a cold punched nut was 1.75-inch, the nut-hole was .3125-in. in diameter, and the length of the piece punched out was only .75-in. Thus, the flow must have taken place through a depth of 1-in., and the shearing through a depth of .75-in. Hence, the surface really shorn was $\pi \times \cdot 3125 \times \cdot 75 = \cdot 736$-sq. ins. in area, and a *measure* of the shearing action is the product of this surface area and the *fluid pressure.*

Again, rails, which have been in use for some time, are found to have acquired an elongated lip at the edge. This is doubtless due to the *flow* of the metal under the great pressures to which the rails are continually subjected.

Note.—The *fluid pressure* for lead is 2844-lbs. per sq. in.

(1).–A chain l-ft. in length and a–sq. in. in sectional area, has one end attached to a weight of W_1-lbs. at rest, and at the other end is a weight of W_2–lbs. moving with a velocity of V–ft. per second away from the first; find the greatest pull upon the chain.

(2).–A vertical prismatic bar of weight W_1 has its upper end fixed, and carries a weight W_2 at the lower end; find the amount and work of the elongation.

(3)–A right cone of weight W rests upon its base: find the amount and work of the compression.

(4).–A tower in the form of a solid of revolution about a vertical axis carries a given *surcharge*, determine the curve of the generating line so that the stress at every point of the tower may be the same.

If the surcharge be zero, shew that the height of the tower becomes infinite but that its volume remains finite.

(5).–Determine the generating curve when the tower in question (4) is hollow, the hollow part being in the form of a right cylinder upon a circular base of given radius.

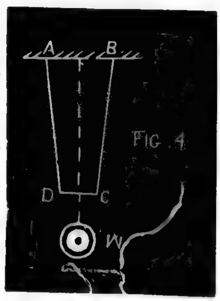
(6).–A heavy vertical bar is fixed at its upper end and carries a given weight at the lower end; determine the form of the bar so that the horizontal sections may be proportionate to the stresses to which they are subjected. (Note. Such a bar is a bar of *uniform strength.*)

(7).–Find the upper and lower sectional areas of a steel shaft of uniform strength 200-ft. in length, which will safely sustain its own weight and 100-tons.

(8).–A vertical elastic rod of natural length L and of which the mass may be neglected, is fixed at its upper end and carries a weight W_1 at the lower end. A weight W_2 falls from a height h upon W_1, find the velocity and extension of the rod at any time t in terms of t.

(9).–Determine the functions F and f in § (4). when P_1 is zero, and also when the rod is perfectly free, *i.e.*, when $P_o=0$, and $P_1=0$.

(10).–An elastic lamina $ABCD$, of natural length l, has its upper edge AB $(2.a)$ fixed and hangs vertically; if a weight W is suspended from the lower edge CD $(2b)$, shew that, neglecting the weight of the lamina, the consequent elongation $=l-\frac{W}{2.E}.\frac{l}{a-b}.\,log.\frac{b}{a}$. If an additional weight is placed upon W and then suddenly removed, shew that the oscillation set up is isochronous and that the time of a complete oscillation



$$=\pi.\left\{\frac{l.\,log.\frac{a}{b}}{2.g.E.(a-b)}\right\}^{\frac{1}{2}}$$

Examine the case when $a=b$., and also when $b=0$.

(11).–A wire of diar. D moves with a velocity V under the action of a force Q in the direction of motion, through a die, and on leaving the die the diar. and velocity are d and v, respectively; determine the mean velocity of the wire and the total pressure upon the die, f being the coeff. of friction.

(12).–A metallic mass rests upon the end CD of a strong cylinder of radius R and fills up a space of depth D. A hole of radius r is made at the centre of the face CD, through which the mass will *flow* when a sufficiently intense pressure is exerted by the piston. Suppose that the mass within the cylinder is compressed to a thickness $DO=x$, the corresponding length of the jet, KE, being y, and assume, (1),–that the specific weight of the mass remains the same, (2),–that the cylindrical portion $EFGH$ is *gradually* transformed into $NMPLKQN$, of which the part $PMNQ$ is cylindrical, while the diameter of the part $PLKQ$ *gradually* increases from the face of the cylinder to KL ($=EF$), at the end of the jet, (3),–that the diminution of the diameter of the cylindrical portion $PMNQ$ is directly proportional to the said diameter.

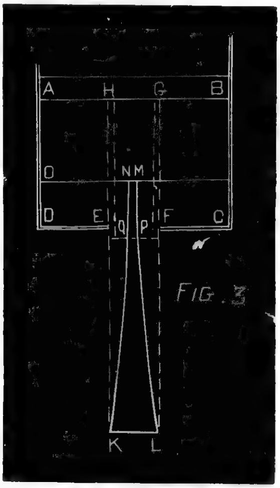


FIG. 3

Shew that $z=r.\left(\frac{x}{D}\right)^n=r.\left(1-\frac{r^2}{R^2}\cdot\frac{y}{D}\right)^n$ where z is the radius of the cylinder $PMNQ$, and $n=\frac{R^2-r^2}{r^2}$

CHAPTER II.

THE EQUILIBRIUM AND STRENGTH OF BEAMS.

Note.–In this chapter it is assumed that all forces act in one and the same plane, and that the deformations are so small as to make no sensible alteration either in the forces or in their relative positions.

(1). – *Equilibrium of Beams.*– A beam is a bar of somewhat considerable scantling, supported at two points, and acted upon by forces perpendicular or oblique to the direction of its length.

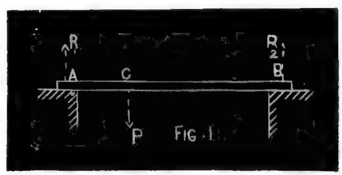


FIG. 1.

Case I.–AB is a beam resting upon two supports in the same horizonta plane. The reactions R_1 and R_2 at the points of support are vertical, and the resultant P of the remaining external forces must also act vertically in an opposite direction at some point C. According to the principle of the lever, $R_1 = P.\frac{BC}{AB}$, $R_2 = P.\frac{AC}{AB}$, and $R_1 + R_2 = P$.

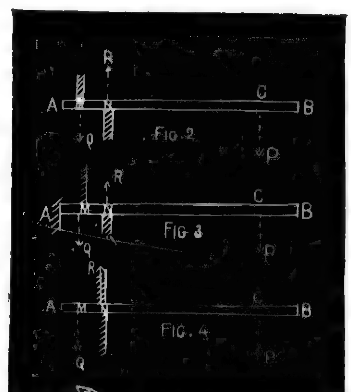


FIG. 2.
FIG. 3.
FIG. 4.

Case II.–AB is a beam supported or *fixed* at one end. Such a support tends to prevent any deviation from the straight in that portion of the beam, and the less the deviation the more perfect is the fixture.

The ends may be fixed by means of two props, Fig. 1, or by allowing it to rest upon one prop and preventing upward motion by a ledge, Fig. 2, or by building it into a wall, Fig. 3.

In any case it may be assumed that the effect of the fixture, whether perfect or imperfect, is to develop two unequal forces, Q and R, acting

in opposite directions at points M and N. These two forces are equivalent to a left-handed couple $(Q,-Q)$ the moment of which is $Q.MN$, and to a single force $R-Q$ at N. Hence $R-Q$ must $= P$.

Case III.—AB is an inclined beam supported at A, and resting upon a smooth vertical surface at B.

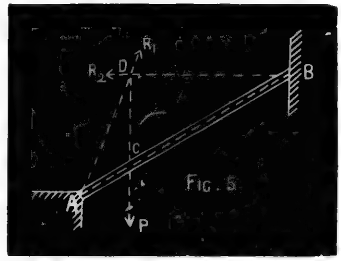


FIG. 5.

The vertical weight P, acting at the point C, is the resultant load upon AB. Let the direction of P meet the horizontal line of reaction at B in the point D.

The beam is kept in equilibrium by the weight P, the reaction R_1 at A, and the reaction R_2 at B. Now the two forces R_2 and P meet at D, so that the force R_1 must also pass through D.

$$\text{Hence } R_1 = P.\frac{1}{cos.ADC}, \text{ and } R_2 = P.tan ADC.$$

Note.—The same principles hold if the beam in Cases I and II is inclined, and also whatever may be the directions of the forces P and R_2, in Case III.

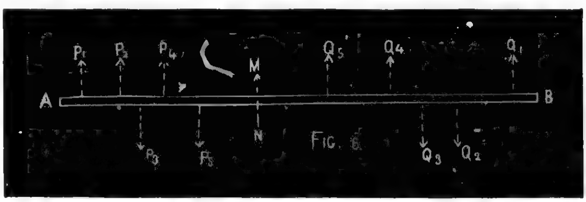
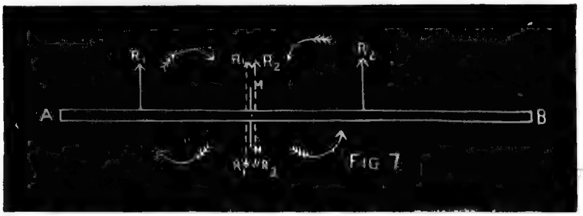


FIG. 6.

Case IV.—*In general*, let the beam AB be in equilibrium under the action of any number of forces $P_1, P_2, P_3, \ldots\ldots Q_1, Q_2, Q_3, \ldots\ldots$ of which the magnitudes and points of application are given, and which act at right angles to the length of the beam. Suppose the beam to be divided into two segments by an imaginary plane MN. Since the whole beam is in equilibrium, each of the segments must also be in equilibrium. Consider the segment AMN.

It is kept in equilibrium by the forces $P_1, P_2, P_3 \ldots\ldots$ and by the reaction of the segment BMN upon the segment AMN, at the plane MN; call this reaction E_1. The forces $P_1, P_2, P_3 \ldots\ldots$ are equivalent to a single resultant R_1, acting at a point distant r_1 from MN. Also, without affecting the equilibrium, two forces, each equal and parallel to R_1, but opposite to one another in direction, may be applied to the

segment AMN, at the plane MN, and the three equal forces are then equivalent to a single force R_1 at MN, and a couple $(R_1,-R_1,)$ of which the moment is $R_1.r_1$.



Thus, the external forces upon AMN are reducible to a single force R_1 at MN, and a couple $(R_1, -R_1)$. These must be balanced by E_1, and therefore E_1 is equivalent to a single force $-R_1$ at MN and a a couple $(-R_1, R_1)$.

In the same manner, the external forces upon the segment BMN are reducible to a single force R_2, at MN, and a couple $(R_2,-R_2)$, of which the moment is $R_2.r_2$. These again must be balanced by E_2, the reaction of the segment AMN upon the segment BMN.

Now E_1 and E_2, evidently neutralise each other, so that the force R_1, and the couple $(R_1,-R_1)$, must neutralise the force R_2, and the couple $(R_2,-R_2)$. Hence the force R_1, and the couple $(R_1,-R_1)$, are respectively equal but opposite in effect to the force R_2, and the couple $(R_2,-R_2)$, *i.e.*, $R_1 = R_2$, and $R_1.\ r_1 = R_2.r_2$, $\therefore r_1 = r_2$.

The force R_1 tends to make the segment AMN slide over the segment BMN, at the plane MN, and is called the *Shearing Force* with respect to that plane. It is equal to the algebraic sum of the forces to the *left* of MN, $= P_1 + P_2 + P_3 + \ldots\ldots = \Sigma\ (P)$.

So $R_2 = Q_1 - Q_2 - Q_3 + \ldots\ldots = \Sigma\ (Q)$, is the algebraic sum of the forces on the *right* of MN, and is the force which tends to make the segment BMN slide over the segment AMN, at the plane MN. R_2 is therefore the *Shearing Force* with respect to MN, and is equal to R_1, in magnitude, but acts in an opposite direction.

Again, let $p_1, p_2, p_3, \ldots\ldots q_1, q_2, q_3 \ldots\ldots$ be respectively the distances of the points of application of $P_1, P_2, P_3 \ldots\ldots Q_1, Q_2, Q_3 \ldots\ldots$ from MN.

Then $R_1.r_1 =$ the algebraic sum of the moments about MN of all the forces on the *left* of $MN = P_1.\ p_1 + P_2.\ p_2, + P_3.\ p_3 + \ldots\ldots = \Sigma(P.p)$, is the moment of the couple $(R_1,-R_1)$.

This couple tends to bend the beam at the plane MN, and its moment is called the *Bending Moment* with respect to MN, of all the forces on the *left* of MN.

So $R_2 . r_2 =$ the algebraic sum of the moments about MN of all the forces on the *right* of MN, $= Q_1 . q_1 - Q_2 . q_2 - \ldots\ldots = \Sigma (Q.q)$, is the *Bending Moment* with respect to MN, of all the forces on the *right* of MN, and is equal but opposite in effect to $R_1 . r_1$.

It is seen that the Shearing Force and Bending Moment *change sign* on passing from one side of MN to the other, so that to define them *absolutely* it is necessary to specify the segment under consideration.

Remark.—The reaction E_1 has been shewn to be equivalent to the force R_1, and the couple $(-R_1, R_1)$. The Moment of this couple may be called the *Elastic Moment*, the *Moment of Resistance*, or the *Moment of Inflexibility*, and is equal in magnitude, but opposite in effect, to the corresponding Bending Moment of the external forces.

(2).–*Examples of Shearing Forces and Bending Moments.*– In each of the following examples the beam is horizontal and of length l.

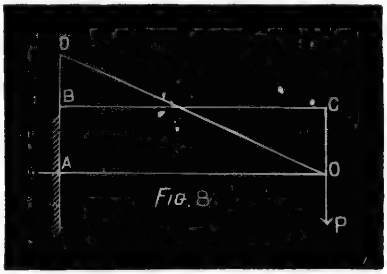


Fig. 8

Ex. 1.–The beam OA, Fig. 8, is fixed at A and carries a weight P at O.

The *Shearing Force* (S) at every point of the beam is evidently constant, and equal to P.

Upon the verticals through A and O take AB and OC, each equal or proportional to P; join BC. The vertical distance between any point of the beam and the line BC represents the shearing force at that point.

Again, the *Bending Moment* (M), at any point of the beam distant x from O, is $P.x$; it is nil at O, and $P.l.$ at A.

Upon the vertical through A take AD, equal or proportional to $P.l$; join DO. The vertical distance between any point of the beam and the line DO represents the bending moment at that point.

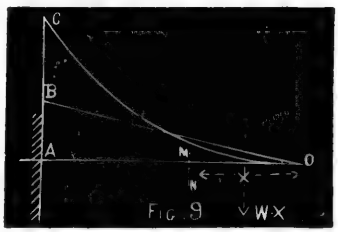


Fig. 9

Ex. 2.–The beam OA, Fig. 9, is fixed at A, and carries a uniformly distributed load, of intensity w per unit of length.

The resultant force on the right of a vertical plane MN distant x from O is $w.x.$, and acts half way between O and MN.

The *Shearing Force* (S) at MN is therefore $w.x$; it is nil at O and $w.l$ at A. Upon the vertical through A, take AB, equal or proportional to $w.l$; join BO. The vertical distance between any point of the beam and the line BO represents the shearing force at that point.

Again, the *Bending Moment* (M) at MN is $w.x.\frac{x}{2} = \frac{w.x^2}{2}$; it is nil at O and $\frac{w.l^2}{2}$ at A. Upon the vertical through A, take AC, equal or proportional to $\frac{w.l^2}{2}$.

The bending moment at any point of the beam is represented by the vertical distance between that point and a parabola CO having its vertex at O and its axis vertical.

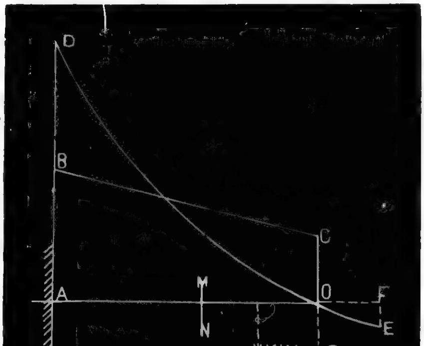


FIG. 10

Ex. 3.–The beam OA, Fig. 10, is fixed at A and carries a single weight P at O, together with a uniformly distributed load of intensity w per unit of length.

The *Shearing Force* (S) at a plane MN, distant x from O, is evidently $P + w.x$; it is P at O and $P + w.l$ at A.

Upon the verticals through O and A, take OC equal or proportional to P, and AB equal or proportional to $w.l + P$; join BC. The vertical distance between any point of the beam and the line BC represents the shearing force at that point.

Again, the *Bending Moment* (M) at MN is evidently $w.x.\frac{x}{2} + P.x = \frac{w.x^2}{2} + P.x$; it is nil at O and $\frac{w.l^2}{2} + P.l$ at A.

Upon the vertical through A, take AD equal or proportional to $\frac{w.l^2}{2} + P.l$. The bending moment at any point of the beam is represented by the vertical distance between that point and a parabola DOE, having its axis EF vertical and its vertex at a point E, where $OF = \frac{P}{w}$ and EF is equal or proportional to $\frac{P^2}{2.w}$

Note.—The ordinates of the line BC are equal to the algebraic sum of the corresponding ordinates of the straight lines BC and BO in Exs. 1 and 2. Also, the ordinates of the curve DO are equal to the algebraic sum of the corresponding ordinates of the line DO in Ex. 1, and the curve CO in Ex. 2. Hence, the same conclusions, as in Ex. 3, are arrived at by treating the weight P and the load $w.l$ independently, and then superposing the respective results.

Ex. 4.–The beam OA, Fig. 11, rests upon two supports at O and A, and carries a weight P at a point B, dividing the beam into the two segments OB, BA, of which the lengths are a and b respectively.

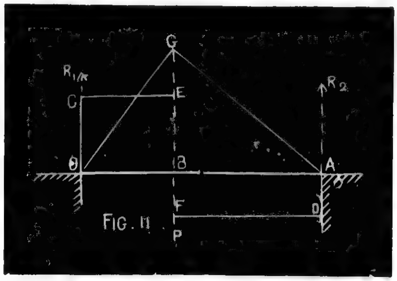


FIG. 11

The reactions R_1, R_2 at O and A are vertical, and according to the principle of the lever,

$$R_1 = P.\frac{b}{l}, \text{ and } R_2 = P.\frac{a}{l}$$

The *Shearing Force* (S) at every point between O and B is constant and equal to $R_1 = P.\frac{b}{l}$. On passing B, the shearing force (S) changes sign, and its value at every point between B and A is constant and equal to $R_1 - P = -P.\frac{a}{l} = -R_2$. Upon the verticals through O, B and A take OC, BE, each equal or proportional to $\frac{P.b}{l}$, and BF, AD, each equal or proportional to $\frac{P.a}{l}$; join CE and DF. The shearing force at any point of the beam is represented by the vertical distance between that point and the broken line $CEFD$.

Again, the *Bending Moment* (M) at any point between O and B, distant x from O, is $R_1.x = P.\frac{b}{l}\,x$; it is nil at O and $P.\frac{a\,b}{l}$ at B.

The *Bending Moment* (M) at any point between B and A, distant x from O is $R_1\,x - P.(x-a) = P.\frac{a}{l}.(l-x)$; it is $P.\frac{a.b}{l}$ at B and nil at A.

$P\frac{bx}{l} - Px + Pa$

$P\left(\frac{b-x}{l}\right) + Pa$

Upon the vertical through B take BG equal or proportional to $P.\frac{a.b}{l}$; join OG and AG. The bending moment at any point of the beam is represented by the vertical distance between that point and the line OGA.

$-Px\frac{a}{l} + Pa$

$\frac{Pa}{l}(l-x)$

Cor.–If P be at the centre of the beam, $\therefore S=\frac{P}{2}$, and M at the centre $=\frac{P.l}{4}$. $a=\frac{l}{2}$ $x=\frac{l}{2}$

Ex. 5.–The beam OA, Fig. 12, rests upon two supports at O and A, and carries a uniformly distributed load of intensity w per unit of length.

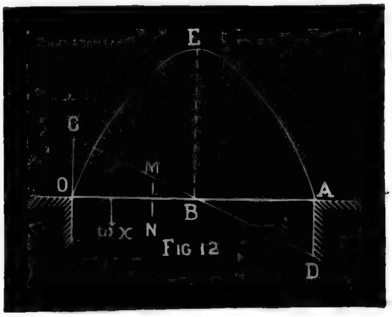


FIG 12

The reactions at O and A are each equal to $\frac{w.l}{2}$.

The resultant force between O and a plane MN distant x from O, is $w.x$, and acts half-way between O and MN. The *Shearing Force* (S) at MN is therefore $\frac{w.l}{2}-w.x$; it is $\frac{w.l}{2}$ at O, nil at the middle point B, and $-\frac{w.l}{2}$ at A. Upon the verticals through O and A, take OC and AD, each equal or proportional to $\frac{w.l}{2}$; join CD. The shearing force at any point of the beam is represented by the vertical distance between that point and the line CD.

Again, the *Bending Moment* (M) at MN is $\frac{w.l}{2}.x-w.x.\frac{x}{2}$ $=\frac{w.l}{2}.x-\frac{w.x^2}{2}$; it is nil at O and at A; it is a maximum and equal to $\frac{w.l^2}{8}$ at the middle point B. Upon the vertical through B take BE, equal or proportional to $\frac{w.l^2}{8}$. The bending moment at any point of the beam is represented by the vertical distance between that point and a parabola OEA having its vertex at E and its axis vertical.

Cor. 1.–The shearing force is a minimum and zero at the centre, a maximum and $\frac{w.l}{2}$ at the ends, and increases uniformly with the distance from the centre.

Cor. 2.–The bending moment is a minimum and zero at the ends, a maximum and $\frac{w.l^2}{8}$ at the centre, and diminishes as the distance from the centre increases.

Ex. 6.–The beam OA, Fig. 13, rests upon two supports at O and A and carries a weight P at a point B, together with a uniformly distributed load of intensity w per unit of length.

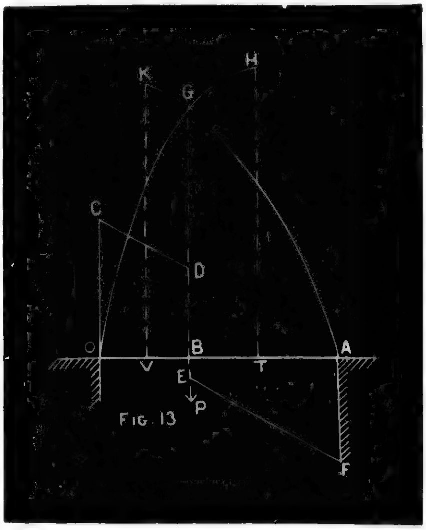


FIG. 13

Let the lengths of the segments OB, BA, be a and b, respectively.

The reactions R_1 at O, and R_2 at A, are vertical, and according to the principle of the lever, $R_1 = P.\frac{b}{l} + \frac{w.l}{2}$, and

$R_2 = P.\frac{a}{l} + \frac{w.l}{2}$.

The *Shearing Force* (S) at any vertical plane, between O and B, distant x from O, is $R_1 - w.x = P.\frac{b}{l} + \frac{w.l}{2} - w.x$;

it is $\frac{P.b}{l} + \frac{w.l}{2}$ at O, and $\frac{P.b}{l} + \frac{w.l}{2} - w.a$ at B.

The *Shearing Force* (S) at any plane between B and A, distant x from O, is $R_1 - P - w.x = P.\frac{b}{l} + \frac{w.l}{2} - P - w.x = \frac{w.l}{2} - P.\frac{a}{l} - w.x$; it is $\frac{w.l}{2} - \frac{P.a}{l} - w.a$ at B, and $-\frac{P.a}{l} - \frac{w.l}{2}$ at A. Upon the verticals through O, B and A, take OC equal or proportional to $\frac{P.b}{l} + \frac{w.l}{2}$, BD equal or proportional to $\frac{P.b}{l} + \frac{w.l}{2} - w.a$, BE equal or proportional to $\frac{w.l}{2} - \frac{P.a}{l} - w.a$, and AF equal or proportional to $\frac{w.l}{2} - \frac{P.a}{l}$; join CD and EF. The shearing force at any point of the beam is represented by the vertical distance between that point and the broken line $CDEF$.

Note.–If $\frac{w.l}{2} > \frac{P.a}{l} + w.a$, BE is positive, and therefore E is vertically *above B*.

Again, the *Bending Moment* (M) at any point between O and B is $\left(P.\frac{b}{l}+\frac{w.l}{2}\right).x-\frac{w.x^2}{2}$; it is nil at O and $\left(P.\frac{b}{l}+\frac{w.l}{2}\right).a-\frac{w.a^2}{2}$ at B.

The bending moment (M) at any point between B and A, distant x from O, is $\left(P.\frac{b}{l}+\frac{w.l}{2}\right).x-\frac{w.x^2}{2}-P.(x-a)$

$=\left(-P.\frac{a}{l}+\frac{w.l}{2}\right).x-\frac{w.x^2}{2}+P.a.$; it is $\left(P.\frac{b}{l}+\frac{w.l}{2}\right)a-\frac{w.a^2}{2}$ at B, and nil at A.

Upon the vertical through B take $B\,G$, equal or proportional to $\left(P.\frac{b}{l}+\frac{w.l}{2}\right).a-\frac{w.a^2}{2}$. The bending moment at any point of the beam between O and B is represented by the vertical distance between that point and a parabola OGH, having its axis HT vertical and its vertex at a point H where $OT=\frac{1}{w}\left(P.\frac{b}{l}+\frac{w\,l}{2}\right)$, and HT is equal or proportional to $\frac{1}{2.w}.\left(P.\frac{b}{l}+\frac{w.l}{2}\right)^2$ The bending moment at any point between B and A is represented by the vertical distance between that point and a parabola AGK having its axis $K\,V$ vertical and its vertex at a point K, where $OV=\frac{1}{w}.\left(-P.\frac{a}{l}+\frac{w.l}{2}\right)$ and KV is equal or proportional to $\frac{1}{2.w}.\left(-P.\frac{a}{l}+\frac{w.l}{2}\right)^2+P.a$

Cor.–If the weight P is at the centre $\therefore S=\frac{P}{2}$, and M, at the centre, $=\frac{P.l}{4}+\frac{w.l^2}{8}$.

Note.–The ordinates of the lines CD and EF are equal to the algebraic sum of the corresponding ordinates of the lines CE, FD in Ex. 4 and the line CD in Ex. 5. Also, the ordinates of the curves OG, AG, are equal to the algebraic sum of the corresponding ordinates of the lines OG, AG in Ex. 4 and the curve OEA in Ex. 5. Hence, the same conclusions, as in Ex. 6, are arrived at by treating the weight P and the load $w.l$ independently, and then superposing the respective results.

Ex. 7.–In fine, a beam, however loaded, may be similarly treated, remembering that if the load changes *abruptly* at different points, the portions of the beam between these points of discontinuity are to be dealt with separately. For example:–the beam OA, Fig. 14, rests upon two supports at O and A, and carries three weights P_1, P_2, P_3, at points C, D, E, of which the distances from O are p_1, p_2, p_3, respectively. A

point B divides OA into segments $OB = a$, and $BA = b$, which are uniformly loaded with weights of intensities w_1 and w_2 per unit of length, respectively. The reactions R_1 and R_2 at O and A, are vertical and according to the principle of the lever:—

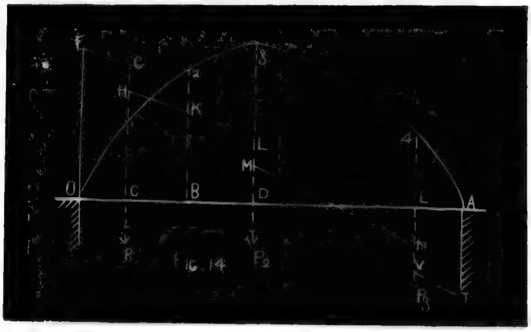


FIG. 14

$$R_1.l = P_1.(l - p_1) + P_2.(l - p_2) + P_3.(l - p_3) + w.a.\left(\frac{a}{2} + b\right) + \frac{w.b^2}{2}$$

and $R_2.l = P_1.p_1 + P_2.p_2 + P_3.p_3 + \dfrac{w.a^2}{2} + w.b.\left(\dfrac{b}{2} + a\right)$

To represent the *Shearing Force* at different points of the beam, graphically :—

Upon the verticals through O, C, B, D, E, A, take, OF, CG, CH, BK, DL, DM, EN, EV, and AT, respectively equal or proportional to,

$$R_1,\ R_1 - w_1.p_1,\ R_1 - w_1.p_1 - P_1,\ R_1 - w_1.a - P_1,$$
$$R_1 - w_1.a - P_1 - w_2.(p_2 - a),\ R_1 - w_1.a_1 - P_1 - w_2(p_2 - a) - P_2,$$
$$R_1 - w_1.a - P_1 - w_2.(p_3 - a) - P_2,\ R_1 - w_1.a - P_1 - w_2(p_3 - a) - P_2 - P_3,$$
$$\text{and } R_1 - w_1.a - P_1 - w_2.b - P_2 - P_3 = R_2.$$

Join FG, HK, KL, MN, and VT. The shearing force at any point of the beam is represented by the vertical distance between that point and the broken line $FGHKLMNVT$.

To represent the *Bending Moment* (M) at different points of the beam, graphically :—

M at $O = 0$, M at $C = R_1.p_1 - \dfrac{w_1.p_1^2}{2}$, M at $B = R_1.a - \dfrac{w_1.a^2}{2} - P_1.(a - p_1)$

$$M \text{ at } D = R_1.p_2 - w_1.a.\left(p_2 - \frac{a}{2}\right) - w_2.\frac{(p_2 - a)^2}{2} - P_1.(p_2 - p_1)$$

$$M \text{ at } E = R_1.p_3 - w.a.\left(p_3 - \frac{a}{2}\right) - w_2.\frac{(p_3 - a)^2}{2} - P_1.(p_3 - p_1) - P_2.(p_3 - p_2)$$

and M at $A = 0$

Upon the verticals through C, B, D, and E, take $C1$, $B2$, $D3$, and $E4$ respectively equal or proportional to the bending moments at these points.

The bending moment at any point of the beam is represented by the vertical distance between that point and the parabolic arcs 01, 12, 23, 34 and $4A$. The axes of these parabolas are vertical, and the positions of the vertices may be easily found from the several equations.

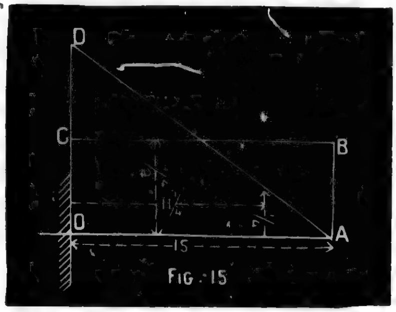


FIG. 15

Ex. a.–A beam OA, Fig. 15, of which the weight may be neglected, is 15ft. long, is fixed at O, and carries a weight of 80–lbs. at A. Determine the bending moment at a point distant 10-ft. from the free end. Also illustrate the shearing force and bending moment at different points of the beam graphically. The required bending moment is $80 \times 10 = 800$-ft. lbs.

The shearing force is the same at every point of the beam, and equal to 80-lbs. Choose a vertical scale of measurement, so that half an inch represents 160-lbs.

Upon OA, describe a rectangle $OABC$, in which $OC = AB = \frac{1}{2}.''$ The ordinate from every point of BC to AO is $\frac{1}{2},''$ or 80-lbs., and is therefore the shearing force at the foot of such ordinate.

Again, the bending moment at O, is $80 \times 15 = 1200$–ft. lbs. Choose a vertical scale of measurement, so that 1-inch represents 1200-ft. lbs Upon the vertical through O, take $OD =$ 1-inch; join DA. The ordinate from any point of DA to OA is the bending moment at its foot For example, at $11\frac{1}{4}$-ft. from O, the ordinate is $\frac{1}{4}''$, or 300-ft. lbs., and this is equal to 80 $\times 3\frac{3}{4}'$, *i.e.*, the bending moment.

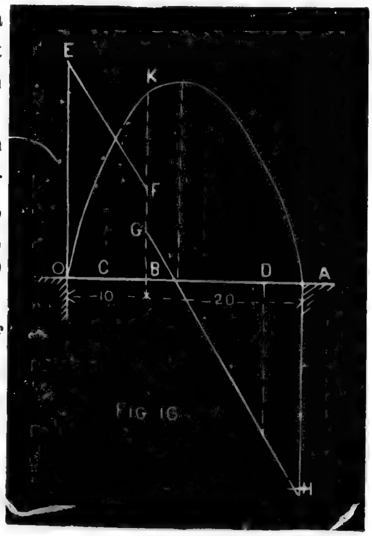


FIG 16

Ex. b.–A beam OA, Fig. 16, of which the weight may be neglected, rests upon two supports at O and A, 30-ft. apart, and carries a uniformly distributed load of 200-lbs. per lineal ft., together with a single weight of 600–lbs. at a point B

dividing the beam into segments $O\,B$, $B\,A$, of which the lengths are 10 and 20-ft. respectively. Determine the shearing force and bending moment at the points C and D, distant 5-ft. from the nearest end. Also, illustrate the shearing force and bending moment at different points of the beam, graphically.

Let R_1, R_2, be the reactions at O and A, respectively.

$\therefore R_1.30 = 600.20 + 200.30.15 = 102000 \;\therefore\; R_1 = 3400$-lbs., and
$R_2 = 200.\;30 + 600 - R_1 = 3200$-lbs

The Shearing Force at $C = 3400 - 200.5 = 2400$-lbs.

" " " $D = 3400 - 200.25 - 600 = -2200$-lbs.

The Bending Moment at $C = 3400.5 - 200.5.\frac{5}{2} = 14{,}500$-ft. lbs.

" " $D = 3400.25 - 200.25.\frac{25}{2} - 600.15 = 13{,}500$-ft. lbs.

Next, considering the segment $O\,B$, the shearing force at O is 3400-lbs., and at B, 1400-lbs.

Considering the segment BA, the shearing force at A is -3200-lbs, and at B, 800-lbs.

Choose a vertical scale of measurement, so that 1-inch represents 3000-lbs. Upon the verticals through O, B, A, take $OE = 1\frac{2''}{15}$, $B\,F = \frac{7''}{15}$, $BG = \frac{4''}{15}$, and $AH = 1\frac{1''}{15}$; join EF and GH. The ordinate from any point of the broken line $EFGH$ to OA is the Shearing Force at its foot. For example, the ordinate at D is $-\frac{11''}{15}$, or -2200-lbs.

Again, the bending moment at B is $3400.10 - 200.10.5 = 24{,}000$-ft. lbs. Choose a vertical scale of measurement, so that 1-inch represents 24,000-ft. lbs. Upon the vertical through B, take $BK =$ 1-inch. Draw the parabolas OK, AK, with their vertices at points determined as in Example (6). The ordinate from any point of the curves OK, AK, is the bending moment at its foot.

For example, at a point 14-ft. from O, the curve ordinate is $1\frac{1}{15}''$, or 25600-ft. lbs., and this is the Bending Moment at the same point being also the greatest for the segment BA. The vertex of AK is, therefore, vertically above the point of which the distance from O is 15-ft.

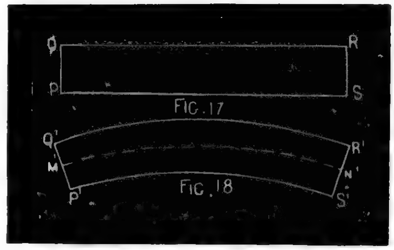


FIG. 17

FIG. 18

(3).–*To determine the Elastic Moment.*–Let the plane of the paper be a plane of symmetry with respect to the beam *PQRS*. If the beam is subjected to the action of external forces in this plane, *PQRS* is bent and assumes a curved form $P'Q'R'S'$. The upper layer of fibres, $Q'R'$, is extended, the lower layer, $P'S'$, is compressed, while of the layers within the beam, those nearer $P'S'$ are compressed, and those nearer $Q'R'$ are extended. Hence, there must be a layer $M'N'$ between $P'S'$ and $Q'R'$ which is neither compressed nor extended. It is called the *Neutral Surface* (or *Cylinder*), and its axis is perpendicular to the plane of flexure. In the present treatise it is proposed to deal with flexure in one plane only, and, in general, it will be found more convenient to refer to $M'N'$ as the *Neutral Line* (or *Axis*).

If a force act upon the beam in the direction of its length, the lower layer $P'S'$, instead of being compressed, may be stretched. In such a case there is no neutral surface *within* the beam, but theoretically it still exists somewhere *without* the beam.

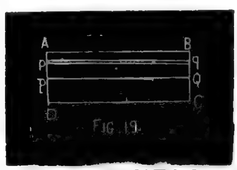


FIG. 19

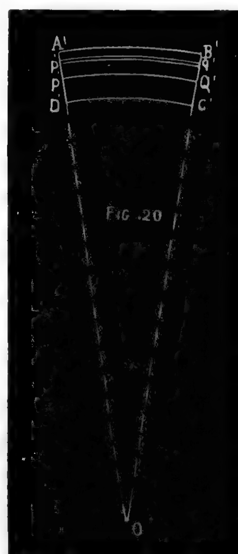


FIG. 20

Let $A\,B\,C\,D$ be an indefinitely small rectangular element of the unstrained beam, and let its length be s. Let $A'B'C'D'$ be the element after deformation by the external forces.

$P'Q'$, the neutral line, being neither compressed nor extended, is unchanged in length and equal to $PQ = s$.

Let the normals at P' and Q' to the neutral line meet in the point O; O is the centre of curvature of $P'Q'$.

Also, as the flexure is very small, the normal planes through OP' and OQ' may be assumed to be perpendicular to all the layers which traverse the corresponding sections of

the beam, so that they must coincide with the planes $A'D'$ and $B'C'$, respectively.

Consider an elementary layer $p'q'$, of length s', sectional area a_1, and distant y_1 from the neutral surface.

Let $OP'=R=OQ'$.

From the similar figures $OP'Q'$ and $O\,p'q'$,

$$\frac{Op'}{OP'}=\frac{p'q'}{P'Q'}, \text{ or } \frac{R+y_1}{R}=\frac{s'}{s}, \text{ and } \therefore \frac{y_1}{R}=\frac{s'-s}{s}$$

Also, if t_1 is the stress along the layer $p'q'$,

$$t_1=E.a_1.\frac{s'-s}{s}=E.\,a_1.\frac{y_1}{R}=\frac{E}{R}.a_1.y_1$$

E being the coefficient of elasticity of the material of the beam.

So, if t_2, a_2, y_2, t_3, a_3, y_3,......are respectively the stress, sectional area, and distance from the neutral surface, of the several layers of the element,

$$\therefore t_2=\frac{E}{R}.a_2.y_2,\ t_3=\frac{E}{R}.a_3.y_3,\ldots\ldots$$

The *total* stress along the beam is the *algebraic* sum of all these elementary stresses, $=t_1+t_2+t_3+\ldots\ldots=\frac{E}{R}.(a_1.y_1+a_2.y_2+\ldots\ldots)$

$$=\frac{E}{R}.\,\Sigma\,(a.y)$$

Again, the moment of t_1 about $P'=t_1.y_1=\frac{E}{R}.a_1.y_1^2$

" " t_2 " " $=t_2.y_2=\frac{E}{R}.a_2.y_2^2$

" " t_3 " " $=t_3.y_3=\frac{E}{R}.a_3.y_3^2$

and so on.

Thus, the *Elastic Moment* for the section $A'D'$ = *algebraic* sum of moments of all the elementary stresses in the different layers about

$$P'=t_1.y_1+t_2.y_2+t_3.y_3+\ \ldots.\ =\frac{E}{R}.(a_1.y_1^2+a_2.y_2^2+\ldots\ldots)=\frac{E}{R}.\,\Sigma\,(a.y^2)$$

Now, $\Sigma\,(a.y^2)$ is the *moment of inertia* of the section of the beam through $A'D'$, with respect to a straight line passing through the neutral line and perpendicular to the plane of flexure, *i.e.*, the plane of the paper. It is usually denoted by I, or $A.k^2$, A being the sectional area and k the radius of gyration.

$\therefore$ the elastic moment$=\frac{E}{R}.\,I=\frac{E}{R}.A.k^2$

But the elastic moment is equal and opposite to the bending moment (M), due to the external forces, at the same section.

$$\text{Hence, } \frac{E}{R}.I=\frac{E}{R}.A.\,k^2=M$$

Note.–It is necessary in the above to use the term *algebraic*, as the elementary stresses change in character, and therefore in sign, on passing from one side of the neutral surface to the other.

Cor. 1.–If the resolved part of the external forces in the direction of the length of the beam be *nil*,

$$\text{the total longitudinal stress}=\frac{E}{R}.\,\Sigma(a.y)=0, \text{ and } \therefore\ \Sigma\,(a.y)=0,$$

shewing that P' must be the centre of gravity of the section through $A'D'$. Hence, when the external forces produce no longitudinal stress in the beam, the neutral line is the *locus* of the centres of gravity of all the sections perpendicular to the length of the beam.

Cor. 2.–If t, a, y, be respectively the stress, sectional area, and distance of a fibre from the neutral line,

$$\therefore \frac{E}{R}.a.y=t, \text{ or } \frac{E}{R}.\,y=\frac{t}{a}=\textit{intensity} \text{ of stress}=f_y, \text{ suppose.}$$

$$\therefore \frac{f_y}{y}=\frac{E}{R}, \text{ and } \frac{E}{R}.\,I=M=\frac{f_y}{y}.I$$

Ex.–A timber beam, 6-ins. square, and 20-ft. long, rests upon two supports and is uniformly loaded with a weight of 1000-lbs. per lineal ft. Determine the stress at the centre in a fibre distant 2-ins. from the neutral line.

Also find the central curvature, E being 1,200,000-lbs.

$I=\dfrac{6.6^3}{12}=108$, $M=1000\times10-1000\times5=5000$-ft.lbs$=60,000$-inch-lbs., and $y=2$-ins.

$\therefore$ from the above equations,

$$\frac{1,200,000}{R}.\ 108=60,000=\frac{f_y}{2}.108$$

$\therefore$ R$=2160$-ins.$=180$-ft., and $f_y=1111\frac{1}{9}$–lbs. per sq. in.

Cor. 3.–The beam is strained to the limit of safety when either of the extreme layers $A'B'$,$D'C'$ is strained to the limit of elasticity. In such a case, the least of the values of $\dfrac{f_y}{y}$ for the extreme layers $A'B'$,$D'C'$, is the great-

est consistent with the strength of the beam, and if f_c and c, are the corresponding intensity of stress, and distance from the neutral axis,

$$\therefore \frac{E}{R} . I = M = \frac{f_c}{c} . I$$

Ex.–Compare the strengths of two similarly loaded beams of the same material of equal lengths, and equal sectional areas, the one being round and the other square.

Let r be the radius of the round beam ; f_r the stress in the extreme fibre.
Let a be a side of the square beam ; f_a " " "

$\therefore \pi . r^2 = a^2$; I, for the round bar, $= \frac{\pi . r^4}{4}$, and for the square bar $= \frac{a^4}{12}$

Also, since the beams are similarly loaded, the bending moments at corresponding points are equal.

$$\therefore \frac{f_r}{r} . \frac{\pi . r^4}{4} = M = \frac{f_a}{\frac{a}{2}} . \frac{a^4}{12}, \text{ so that } \frac{f_r}{f_a} = \frac{2}{3} . \frac{a^3}{\pi . r^3} = \frac{2}{3} . \sqrt{\frac{22}{7}} = \sqrt{\frac{88}{63}}$$

Thus, under the same load, the round beam is strained to a greater extent than the square beam, and the latter is the stronger in the ratio of $\sqrt{88}$ to $\sqrt{63}$.

Cor. 4.–The neutral surface is neither stretched nor compressed, so that it is not subjected to any longitudinal stress. But it by no means follows that this surface is wholly free from stress, and it will be subsequently seen that the effect of a shearing force, when it exists, is to stretch and compress the different particles in diagonal directions making angles of 45° with the surface.

Cor. 5.–For a rectangular beam, $I = \frac{b . d^3}{12}$, and $c = \frac{d}{2}$

$$\therefore M = \frac{f}{c} . I = \frac{f}{\frac{d}{2}} . \frac{b . d^3}{12} = \frac{f}{6} . b . d^2$$

If the beam be fixed at one end, and loaded at the other with a weight W, the maximum bending moment $= W . l$

If the beam be fixed at one end, and loaded uniformly with a weight $w . l = W$, the maximum bending moment $= \frac{w . l^2}{2} = \frac{W . l}{2}$

If the beam rest upon two supports, and carry a weight W at the centre, the maximum bending moment $= \frac{W . l}{4}$

If the beam rest upon two supports, and carry a uniformly distributed load of $w.l = W$, the maximum bending moment $= \frac{w.l^2}{8} = \frac{W.l}{8}$

Hence, in the first case, $W = \frac{f}{6} \cdot \frac{b.d^2}{l}$

" " second " $W = \frac{f}{6} \cdot 2.\frac{b.d^2}{l}$

" " third " $W = \frac{f}{6} \cdot 4.\frac{b.d^2}{l}$

" " fourth " $W = \frac{f}{6} \cdot 8.\frac{b.d^2}{l}$

In general, $W = \frac{f}{6} \cdot q.\frac{bd^2}{l}$

q being some co-efficient depending upon the manner of the loading.

Now, if the laws of elasticity held true up to the point of rupture, these equations would give the *Breaking Weights* (W), corresponding to different ultimate unit stresses (f), but the values thus derived, differ widely from the results of experiment. It is usual to determine the Breaking Weight (W) of a rectangular beam from the formula $W = C.\frac{b.d^2}{l}$, where C is a constant which depends both upon the manner of the loading and the nature of the material, and is called the *Coefficient of Rupture.*

The preceding equations, however, may be evidently employed to determine the breaking weights in the several cases by making $\frac{f}{6}.q = C$.

The values of C for iron, steel and timber beams, supported at the two ends and loaded in the centre, are given in the Tables at the end of Chapter I.

The corresponding value of f is obtained from the equation $\frac{f}{6}.4 = C$; $\therefore f = \frac{3}{2}.C$

Ex.–Determine the central breaking weight of a red pine beam, 10-ins. deep, 6-ins. wide, and resting upon two supports, 20-ft. apart.

The value of C for red pine is about 5700.

$\therefore$ the Breaking Weight $= W = 5700.\frac{6.10^2}{20 \times 12} = 14{,}250$-lbs.

(4).–*Moments of Inertia.*–The reader is recommended to commit to memory the following Table of Moments of Inertia, in which m is the mass of a unit of area.

The Moment of Inertia I,

(a).–Of a rectangle, of which the sides are b and d, about an axis through the centre perpendicular to the side d $\quad = m.\dfrac{b.d^3}{12}$

(b).–Of a circle, of which the radius is r, about a diameter $= m.\dfrac{\pi.r^4}{4}$

(c).–Of an ellipse, of which the major and minor axes are $2.b$ and $2.d$ respectively, about the major axis $\quad = m.\dfrac{\pi.b.d^3}{4}$

" " about the minor axis $\quad = m.\dfrac{\pi.b^3.d}{4}$

(d).–Of an annulus, of which the internal and external radii are r and r' respectively, about a diameter $\quad = m.\pi.\dfrac{r'^4 - r^4}{4}$

Note.–Let $t = r' - r$, be the thickness of the annulus, and suppose that t is small compared with r.

$\therefore r'^4 - r^4 = (r+t)^4 - r^4 = 4.r^3.t + 12.r^2.t^2 + 4.r.t^3 + t^4 = 4.r^3.t$, approximately.

$$\therefore I = m.\pi.\frac{r'^4 - r^4}{4} = m.\pi.r^3.t.$$

Ex.–A standpipe section 33-ft. in length, and weighing 5720 lbs., is placed upon two supports in the same horizontal plane, 30-ft. apart. The internal diameter of the pipe is 30-ins., and its thickness $\frac{1}{2}$ an inch. Determine the additional uniformly distributed load which the pipe can carry between the bearings, so that the stress in the metal may nowhere exceed 2-tons per sq. in.

Let W be the required load in lbs.

The weight of the pipe between the bearings $= \dfrac{30}{33}.5720 = 5200$-lbs.

Thus, the total distributed weight between the bearings $= (W + 5200$-lbs.)

$$\text{Now } M = \frac{f_c}{c}.I,$$

and the stress in the metal is necessarily greatest at the central section.

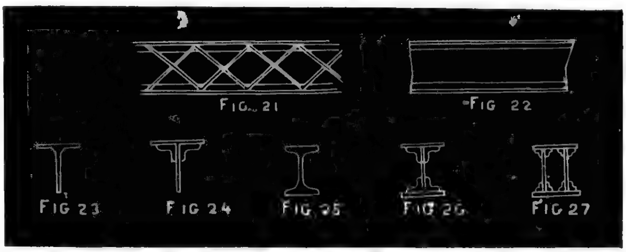
M, at the centre, $= \dfrac{W + 5200}{8}.30.12$-inch lbs.; $f_c = 2 \times 2240$–lbs., and

$\dfrac{I}{c} = \pi.r^2.t = \dfrac{22}{7}.15^2.\dfrac{1}{2}$

$$\therefore \frac{W+5200}{8}.30.12=2.2240.\frac{22}{7}.15^2.\tfrac{1}{2}=72000\times 22, \text{and} \therefore W=30{,}000 \text{ lbs.}$$

(5).–*Flanged Girders, &c.*–Beams subjected to forces, of which the lines of action are at right angles to the direction of their length, are usually termed *Girders;* a *Semi-Girder*, or *Cantilever*, is a girder with one end fixed and the other free.

It has been shewn that the stress in the different layers of a beam increases with the distance from the neutral surface, so that the most effective distribution of the material is made by withdrawing it from the neighbourhood of the neutral surface, and concentrating it in those parts which are liable to be more severely strained. This consideration has led to the introduction of *Flanged Girders,* i.e., girders consisting of *one* or *two Flanges* (or *Tables*), united to *one* or *two webs,* and designated *Single-Webbed* or *Double-Webbed* (*Tubular*) accordingly.



The web may be open like lattice-work (Fig. 21), or closed and continuous (Fig. 22).

The principal sections adopted for Flanged Girders are:—

The *Tee* (Figs. 23 and 24), the I or *Double-Tee* (Figs. 25 and 26), the *Tubular* or *Box* (Fig. 27).

(6).–*Classification of Flanged Girders.*–Generally speaking, Flanged Girders may be divided into two classes, viz:—

I.–Girders with Horizontal Flanges.–In these the flanges can only convey horizontal stresses, and the shearing force, which is vertical, must be wholly transmitted to the flanges through the medium of the web.

If the web be open, or lattice-work, the flange stresses are transmitted through the lattices.

If the web be continuous, the distribution of stress, arising from the transmission of the shearing force is indeterminate, and may lie in certain curves; but the stress at every point is resolvable into vertical

and horizontal components. Thus, the portion of the web adjoining the flanges bears a part of the horizontal stresses, and aids the flanges to an extent dependent upon its thickness.

With a thin web this aid is so trifling in amount that it may be disregarded without serious error.

II.–Girders with one or both flanges curved.–In these the shearing stress is borne in part by the flanges, so that the web has less duty to perform, and requires a proportionately less sectional area.

(7).–*Equilibrium of Flanged Girders.*–

A B is a girder in equilibrium under the action of external forces, and has its upper flange compressed and its lower flange extended. Suppose the girder to be divided into two segments by an imaginary vertical plane *MN*.

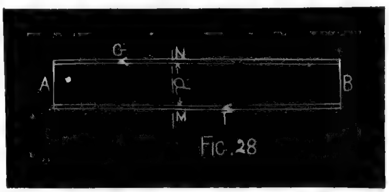


FIG. 28

Consider the segment *AMN*. It is kept in equilibrium by the external forces on the left of *MN*, by the compressive flange stress at $N\ (=C)$, by the tensile flange stress at $M\ (=T)$, and by the vertical and horizontal web stresses along *MN*. The horizontal web stresses may be neglected if the web is thin, while the vertical web stresses pass through *M* and *N*, and consequently have no moments about these points.

Let d be the *effective* depth of the girder, i.e., the distance between the points of application of the flange stresses in the plane *MN*.

Take moments about *M* and *N*, successively.

$\therefore C.d =$ the algebraic sum of the moments about *M* of the external forces upon $AMN =$ the bending moment at $MN = M$.

So, $T.d = M$, $\therefore C.d = M = T.d$, and $C = T$.

Hence, the flange stresses at any vertical section of a girder are equal in magnitude but opposite in kind. The flange stress, whether compressive or tensile, may be denoted by F.

Ex.–A flanged girder, of which the effective depth is 10-ft., rests upon two supports 80-ft. apart, and carries a uniformly distributed load of 2500-lbs. per lineal foot. Determine the flange stress at 10-ft. from the end, and find the area of the flange at this point, so that the unit-stress in the metal may not exceed 10,000-lbs. per sq. in.

The vertical reaction at each support $= \dfrac{80 \times 2500}{2} = 100{,}000$-lbs.

$\therefore F.10 = M = 100,000 \times 10 - 2500 \times 10 \times 5 = 875,000$-ft. lbs.

$\therefore F = 87,500$-lbs.

The required area $= \dfrac{87,500}{10,000} = 8.75$-sq. ins.

Cor. 1.–$F.d = M = \dfrac{E}{R}.I = \dfrac{f_y}{y}.I$

Cor. 2.–At any vertical section of a girder,

let a_1, a_2, be the sectional areas of the lower and upper flanges, respectively.
" f_1, f_2, " " unit stresses in " " " " "

$$\therefore a_1.f_1 = F = a_2.f_2$$

and the sectional areas are inversely proportional to the unit stresses.

Ex.–At a given vertical section of a flanged girder the sectional area of the top flange is 10-sq. ins., and the corresponding unit stress is 8,000-lbs. per sq. in. Find the sectional area of the lower flange, so that the unit stress in it may not exceed 10,000-lbs. per sq. in.

$a_1.\ 10,000 = F = 10.8,000$, $\therefore a_1 = 8$-sq. ins., and $F = 80,000$-lbs.

Note.–The compressive strength of cast-iron is almost 6 times as great as the tensile strength, and therefore the area of the tension flange of a girder of this material should be about 6 times that of the compression flange.

The formula, $W = C.\dfrac{a.d}{l}$, is often employed to determine the strength of a cast or wrought-iron girder which rests upon two supports *l-inches* apart, d being its depth in inches, and a the net sectional area of the bottom flange in sq.-ins. C is a constant to be determined by experiment. Its average value for *cast-iron* is 24 or 26, according as the girder is cast on its side or with its bottom flange upwards. An average value of C for *wrought-iron* is 80.

Cor. 3.–A girder, with horizontal flanges, of length l and depth d, rests upon two supports, and is uniformly loaded with a weight w per unit of length.

The bending moment at a vertical plane distant x from the centre is,

$$M = \frac{w.l}{2}.\left(\frac{l}{2} - x\right) - w.\left(\frac{l}{2} - x\right).\tfrac{1}{2}.\left(\frac{l}{2} - x\right) = \frac{w.l^2}{8}.\left(1 - \frac{4.x^2}{l^2}\right)$$

Also, $M = F.d = a.f.d$, a being the sectional area of either flange at the plane under consideration, and f the corresponding unit stress.

$$\therefore a.f.d = \frac{w.l^2}{8}.\left(1 - \frac{4.x^2}{l^2}\right)$$

Let A be the flange sectional area at the centre,

$$\therefore A.f.d = \frac{w.l^2}{8}$$

$$\text{Hence, } a = A.\left(1 - \frac{4.x^2}{l^2}\right)$$

an expression by which the flange sectional area at any point of the girder may be obtained when the area at the centre is known.

Cor. 4.–F represents indifferently the sum of the horizontal elastic forces either above or below the neutral axis, and is therefore proportional to A, the sectional area of the girder; d is the distance between the centres of resultant stress and is proportional to D, the depth of the girder;

$$\therefore M \propto A.D = C.A.D$$

a form frequently adopted for solid rectangular (*Cor.* 5, § 3), or round girders, but also applicable to other forms.

Remark.–The effective length of a girder may be taken to be the distance from centre to centre of bearings.

The effective depth depends in part upon the character of the web, but in the calculation of flange stresses the following approximate rules are sufficiently accurate for practical purposes :—

If the web is continuous and very thin the effective depth is the full depth of the girder.

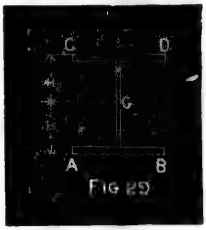
If the web is continuous, and too thick to be neglected, the effective depth is the distance between the inner surfaces of the flanges.

If the web is open, or lattice-work, the effective depth is the vertical distance between the points of attachment of the lattices.

If the flanges are cellular, the effective depth is the distance between the centres of the upper and lower cells.

(8).–*Examples of Moments of Inertia.*

(*a*).–*Double-tee section.*



First, suppose the web to be so thin that it may be disregarded without sensible error.

Let the neutral axis pass through G, the centre of gravity of the section.

Let a_1, a_2, be the sectional areas of the lower and upper flanges, respectively.

Let h_1, h_2, be the distances from G, of the surfaces AB, CD, respectively.

Let $h_1 + h_2 = d$.

$a_1 a_2 d^2 + a_1 a_2 d$ / $(a_1+a_2)^2$; $a_1 a_2 d$ / $(a_1+a_2)^2$

Approximately, $I = a_1 . h_1^2 + a_2 . h_2^2$,

also, $(a_1 + a_2) . h_1 = a_2 . d$, and $(a_1 + a_2) . h_2 = a_1 . d$

$$\therefore I = a_1 . \left(\frac{a_2 . d}{a_1 + a_2}\right)^2 + a_2 . \left(\frac{a_1 . d}{a_1 + a_2}\right)^2 = \frac{a_1 . a_2 . d^2}{a_1 + a_2}$$

Again, if f_1, f_2, respectively, be the unit stresses in the metal of the lower and upper flanges,

$$\therefore M = \frac{f_1}{h_1} . I = f_1 . a_1 . d, \text{ and also } = \frac{f_2}{h_2} . I = f_2 . a_2 . d.$$

Cor.–If $a_1 = a_2 = a$, $\therefore f_1 = f_2 = f$, suppose, and $M = f.a.d.$

Second.–Let the web be too thick to be neglected.

Fig 30

As before, let the neutral axis pass through G, the centre of gravity of the section.

Let a_1, a_2, be the sectional areas of the lower and upper flanges, respectively.

Let a_3, a_4, be the sectional areas of the portions of the web below and above G, respectively.

Let h_1, h_2, be the distances from G, of the surfaces AB, CD, respectively.

Let $h_1 + h_2 = d$.

$$\text{Approximately, } I = a_1 . h_1^2 + a_3 . \left(\frac{h_1^2}{12} + \frac{h_1^2}{4}\right) + a_2 . h_2^2 + a_4 . \left(\frac{h_2^2}{12} + \frac{h_2^2}{4}\right)$$

$(I_h = I_g + Mh^2)$

$$= \left(a_1 + \frac{a_3}{3}\right) . h_1^2 + \left(a_2 + \frac{a_4}{3}\right) . h_2^2$$

Also, $\left(a_1 + \frac{a_3}{2}\right) . h_1 = \left(a_2 + \frac{a_4}{2}\right) . h_2$, and this equation, together with $h_1 + h_2 = d$, will give the values of h_1, h_2; hence the value of I may be determined.

As in the above, $\frac{f_1}{h_1} . I = M = \frac{f_2}{h_2} . I$

Cor. 1.–Let $a_1 = a_2 = A$, and $a_3 = a_4 = \frac{A'}{2}$, $\therefore h_1 = h_2 = \frac{d}{2}$,

and $I = \left(A + \frac{A'}{6}\right) . \frac{d^2}{2}$

$$\therefore M = \frac{2.f}{d} . \left(A + \frac{A'}{6}\right) . \frac{d^2}{2} = f . \left(A + \frac{A'}{6}\right) . d$$

f being the unit stress in either flange.

Thus, the web aids the girder to an extent equivalent to the increase which would be derived by adding *one-sixth* of the web area to each flange.

If the wt of material remain constant M increases with d the depth at the same time, the thickness of the web diminishes to minimum value being limited by certain practical considerations (P 65) Hence it follows that distributing material is most effective when it is concentrated as far as possible from neutral axis

Cor. 2.–The principles of construction require a beam or girder to be designed in such a manner as to be of uniform strength, *i.e.*, equally strained at every point. An exception, however, is usually made in the case of *timber* beams or girders. The fibres of this material are real fibres, and offer the most effective resistance in the direction of their length, so that if they are cut their remaining strength is due only to cohesion with the surrounding material. Besides, there is no economy to be gained by removing a lateral portion, as the waste is of little, if any, practical value.

Ex.–The lower and upper flanges of the section of a girder are 1-in. and $1\frac{1}{2}$-in. thick respectively, and are each 24-ins. wide; the effective depth of the girder is 48-ins., and the web is $\frac{1}{2}$-in. thick. Determine the position of the neutral axis; also find the flange unit stresses when the bending moment at the given section is 250-ft. tons. Use the notation of Cor. 1.

$$\therefore a_1 = 24\text{-sq. ins. } a_2 = 36\text{-sq. ins., and } a_3 + a_4 = 24\text{-sq. ins.}$$

The centre of gravity of the web is at G, half-way between AB and CD.

Thus, $24.\,h_1 + 24.(h_1 - 24) = 36.\,(48 - h_1)$

$$\text{or } h_1 = \frac{192''}{7}, \text{ and } \therefore h_2 = \frac{144''}{7}, \text{ defining the position of } G.$$

$$\text{Again, } a_3 = \frac{192}{7}.\frac{1}{2} = \frac{96}{7}\text{-sq. ins., and } a_4 = \frac{144}{7}.\frac{1}{2} = \frac{72}{7}\text{-sq. ins.}$$

$$\therefore I = \left(24 + \frac{32}{7}\right).\left(\frac{192}{7}\right)^2 + \left(36 + \frac{24}{7}\right).\left(\frac{144}{7}\right)^2 = \frac{267264}{7}$$

Also, $M = 250\text{-ft.-tons} = 3000\text{-inch tons.}$

$$\therefore \frac{7}{192}.f_1.\frac{267264}{7} = 3000 = \frac{7}{144}.f_2.\frac{267264}{7} = f_1.1392 = f_2.1856$$

$$\therefore f_1 = 2\frac{9}{58}\text{-tons per sq. in., and } f_2 = 1\frac{143}{232}\text{-tons per sq. in.}$$

Third.–The value of I for a double-tee section may be more accurately determined as follows:–

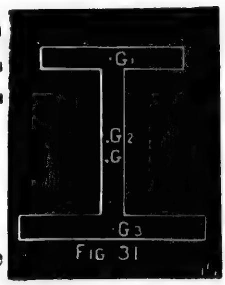


FIG 31

Let the area of the top flange be A_1, and its depth h_1
" " bottom " A_3, " h_3
" " web " A_2, " h_2

Let $A_1 + A_2 + A_3 = A$, and $h_1 + h_2 + h_3 = h$.

Let G be the centre of gravity of the section.
" G_1 " " " top flange
" G_2 " " " web
" G_3 " " " bottom of flange

Let y_1 be the distance of G from the upper edge of the section
" y_2 " " " lower " "
Take moments about G_3.

$$\therefore (A_1 + A_2 + A_3).G\,G_3 = A_1.G_1\,G_3 + A_2.G_2\,G_3.$$

$$= A_1.\left(\frac{h_1}{2} + h_2 + \frac{h_3}{2}\right) + A_2.\left(\frac{h_2}{2} + \frac{h_3}{2}\right)$$

$$\text{or } GG_3 = \frac{A_1.(h_1 + 2.h_2 + h_3) + A_2.(h_2 + h_3)}{2.A}$$

$$\text{So, } GG_2 = \frac{-A_1.(h_1 + h_2) + A_3.(h_2 + h_3)}{2.A}$$

$$\text{and } GG_1 = \frac{A_2.(h_1 + h_2) + A_3.(h_1 + 2.h_2 + h_3)}{2.A}$$

$$\text{Hence, } y_2 = GG_3 + \frac{h_3}{2} = \frac{A_1(h_1+2.h_2+h_3)+A_2(h_2+h_3)}{2.A} + \frac{h}{2} - \frac{h_2}{2} - \frac{h_1}{2}$$

$$\therefore y_2 = \frac{h}{2} - \frac{(h_1 + h_2).(A_1 + A_2 + A_3) - A_1.(h_1 + 2.h_2 + h_3) - A_2.(h_2 + h_3)}{2.A}$$

$$= \frac{h}{2} - \frac{A_3.(h_1 + h_2) - A_1.(h_2 + h_3) - A_2.(h_3 - h_1)}{2.A}$$

$$\text{So, } y_1 = GG_1 + \frac{h_1}{2} = \&c.$$

Again, I, with respect to G,

$$= \frac{A_1.h_1^2}{12} + A_1.G_1\,G^2 + \frac{A_2.h_2^2}{12} + A_2.G_2G^2 + A_3.\frac{h_3^2}{12} + A_3.G_3G^2$$

$$= I_1 + A_1.G_1G^2 + A_2.G_2G^2 + A_3.G_3G^2,\ I_1 \text{ being equal to } \frac{A_1.h_1^2 + A_2.h_2^2 + A_3.h_3^2}{12}$$

$$\therefore I = I_1 + \frac{A_1}{4.\,A^2}.\left\{ A_2.(h_1 + h_2) + A_3.(h_1 + 2.h_2 + h_3) \right\}^2$$
$$+ \frac{A_2}{4.A^2}.\left\{ -A_1.(h_1 + h_2) + A_3.(h_2 + h_3) \right\}^2$$
$$+ \frac{A_3}{4.A^2}.\left\{ A_1.(h_1 + 2.h_2 + h_3) + A_2.(h_2 + h_3) \right\}^2$$

$$= I_1 + \frac{1}{4.A^2}.\left\{ \begin{array}{c} A_1.A_2^2.(h_1+h_2)^2 + 2.A_1.A_2.A_3.(h_1+h_2).(h_1+2.h_2+h_3) \\ +A_1.A_3^2.(h_1+2.h_2+h_3)^2 \\ +A_2.A_1^2.(h_1+h_2)^2 - 2.A_1.A_2.A_3.(h_1+h_2).(h_2+h_3) \\ +A_2.A_3^2.(h_2+h_3)^2 \\ +A_3.A_1^2.(h_1+2.h_2+h_3)^2 + 2.A_1.A_2.A_3.(h_1+2.h_2+h_3).(h_2+h_3) \\ +A_3.A_2^2.(h_2+h_3)^2 \end{array} \right\}$$

$$= I_1 + \frac{1}{4.A^2}.\left\{ \begin{array}{c} A_1.A_2.(h_1+h_2)^2.(A_1+A_2+A_3) - A_1.A_2.A_3.(h_1+h_2)^2 \\ +A_2.A_3.(h_2+h_3)^2.(A_1+A_2+A_3) - A_1.A_2.A_3.(h_2+h_3)^2 \\ -2A_1.A_2.A_3.(h_1+h_2).(h_2+h_3). \\ +(A_3.A_1^2.+A_3^2.A_1).(h_1+2h_2+h_3)^2 \\ +2.A_1.A_2.A_3.(h_1+h_2+h_2)\,(h_1+h_2+h_2+h_3) \end{array} \right\}$$

$$=I_1+\frac{1}{4.A^2}.\left\{\begin{array}{c} A_1.A_2.(h_1+h_2)^2.A+A_2.A_3.(h_2+h_3)^2.A \\ -A_1.A_2.A_3.\,(h_1+2.h_2+h_3)^2 \\ +(A_3A_1^2+A_3^2.A_1).(h_1+2.h_1+h_3)^2 \\ +2.A_1.A_2.A_3.(h_1+2.h_2+h_3)^2 \end{array}\right\}$$

$$=I_1+\frac{1}{4.A^2}.\left\{\begin{array}{c} A_1.A_2.(h_1+h_2)^2.A+A_2.A_3.\,(h_2+h_3)^2.A \\ +A_1.A_3.(h_1+2.h_2+h_3)^2.A \end{array}\right\}$$

Hence, finally,

$$I=\frac{A_1.h_1^2+A_2.h_2^2+A_3.h_3^2}{12}$$

$$+\frac{1}{4.A}.\{A_1.A_2.(h_1+h_2)^2+A_2.A_3.(h_2+h_3)^2+A_1.A_3.(h_1+2h_2+h_3)^2\}$$

Cor. 1.–If h_1, and h_3, are small compared with h_2, put $h_2=h'-\frac{h_1+h_3}{2}$,

$$\therefore y_2=\frac{A_1.2h'+A_2.\left(h'+\frac{h_3-h_1}{2}\right)}{2.A}+\frac{h_3}{2}=\frac{h'}{2}.\frac{2.A_1+A_2}{2.A},\text{ nearly.}$$

$$\text{and } I=\frac{A_1.\,h_1^2+A_2.\left(h'-\frac{h_1+h_3}{2}\right)^2+A_3.h_3^2}{12}$$

$$+\frac{1}{4.A}.\left\{A_1.A_2.\left(h'+\frac{h_1-h_3}{2}\right)^2+A_2.\,A_3.\left(h'+\frac{h_3-h_1}{2}\right)^2+A_3.A_1.4.h'^2\right\}$$

$$=\frac{A_2.\,h'^2}{12}+\frac{1}{4.A}.\left\{A_1.A_2.h'^2+A_2.A_3.h'^2+4A_3.A_1.h'^2\right\}$$

$$\therefore I=h'^2.\left\{\frac{A_2}{12}+\frac{A_1.A_2+A_2.A_3+4.A_3.A_1}{4.A}\right\}.$$

Note.–If A_2 is also very small, as in the case of an open web,

$$\therefore y_2=\frac{h'}{2}.\frac{A_1}{A},\text{ and } I=h'^2.\frac{A_3.A_1}{A},\text{ approximately.}$$

Cor. 2.–Let y_a, y_b, be the distances of G from the upper and lower edges, respectively; let f_a, f_b, be the corresponding ultimate unit stresses.

From the preceding corollary, $y_b=y_2=\frac{y_a+y_b}{2}.\frac{2.A_1+A_2}{2.A}$

$$\text{or, }\frac{A_1+A_2+A_3}{2.A_1+A_2}=\frac{y_a+y_b}{2.y_b}$$

$$\therefore A_3=A_1.\frac{y_a}{y_b}+A_2.\frac{y_a-y_b}{2.y_b}=A_1.\frac{f_a}{f_b}+A_2.\frac{f_a-f_b}{2.f_b}$$

$$\text{Hence, } A_1+A_2+A_3=A_1+A_2+A_1.\frac{f_a}{f_b}+A_2.\frac{f_a-f_b}{2.f_b}$$

$$= \frac{f_a+f_b}{f_b}\cdot\left(A_1+\frac{A_2}{2}\right) = A$$

$$\text{and, } I = h'^2\left\{\frac{A_2}{12} + \frac{A_1.A_2 + A_2.A_3 + 4.A_3.A_1}{4.A}\right\}$$

$$= h'^2.\left\{\frac{4.A_1.A_2 + A_2^2 + 4.A_2.A_3 + 12.A_3.A_1}{12.A}\right\}$$

$$= \frac{h'^2}{12}.\left\{\frac{4.A_1.A_2 + A_2^2 + 4.(A_2+3.A_1).\left(A_1.\frac{f_a}{f_b} + A_2.\frac{f_a-f_b}{2.f_2}\right)}{A}\right\}$$

$$= \frac{h'^2}{12}.\left\{\frac{A_2^2.\left(\frac{2.f_a-f_b}{f_b}\right) + 2.A_1.A_2\left(\frac{2.f_a-f_b}{f_b}\right) + \frac{f_a}{f_b}.(6.A_1.A_2+12.A_1^2)}{\frac{f_a+f_b}{2.f_b}.(2.A_1+A_2)}\right\}$$

$$\therefore I = \frac{h'^2}{6}.\frac{A_2.(2.f_a-f_b)+6.A_1.f_a}{f_a+f_b}$$

$$= \frac{h'}{6}.\frac{y_a}{f_a}.\left\{A_2.(2.f_a-f_b)+6.A_1.f_a\right\}$$

(b).–T section.

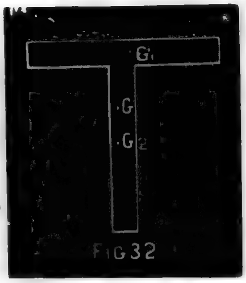


FIG 32

Let the area of the flange be A_1, and its depth h_1
" " web " A_2, " " h_2

Let $A_1 + A_2 = A$, and $h_1 + h_2 = h$.

Let G be the centre of gravity of the section, G_1 of the flange, and G_2 of the web.

Let y_1 be the distance of G from foot of the web.

$$\therefore y_1.(A_1+A_2) = A_1.\left(\frac{h_1}{2}+h_2\right) + A_2.\frac{h_2}{2}$$

$$= \frac{(A_1+A_2)\,(h_1+h_2)}{2} + \frac{A_1.h_2 - A_2.h_1}{2}$$

$$\text{and } y_1 = \frac{h_1+h_2}{2} + \frac{A_1.h_1 - A_2.h_2}{2.\,(A_1+A_2)} = \frac{h}{2} + \frac{A_1.h_1 - A_2.h_2}{2.A}$$

$$\text{Again, } G_1G = \frac{h_1}{2} + h_2 - y_1 = \frac{A_2.h}{2.A}, \text{ and } G_2G = y_1 - \frac{h_2}{2} = \frac{A_1.h}{2.A}$$

$\therefore I$, with respect to a horizontal line through G

$$= A_1.\frac{h_1^2}{12} + A_1.G_1G^2 + A_2.\frac{h_2^2}{12} + A_2.G_2G^2$$

which reduces to $I = \dfrac{A_1.h_1^2 + A_2.h_2^2}{12} + \dfrac{A_1.A_2.h^2}{4.A}$.

Cor. 1.–If h_1 is very small as compared with h_2, put $h_2 = h' - \dfrac{h_1}{2}$

$$\therefore y_1.(A_1+A_2)=A_1.h'+A_2.\left(\frac{h'}{2}-\frac{h_1}{4}\right)=\left(A_1+\frac{A_2}{2}\right).h', \text{ nearly,}$$

$$\text{or, } y_1=\frac{h'}{2}.\frac{2A_1+A_2}{A}$$

$$\text{and } I=\frac{A_1.h_1^2+A_2.\left(h'-\frac{h_1}{2}\right)^2}{12}+\frac{A_1.A_2.\left(h'+\frac{h_1}{2}\right)^2}{4.A}$$

$$=\frac{A_2.h'^2}{12}+\frac{A_1.A_2.h'^2}{4.A}, \text{ nearly,}$$

$$\text{or } I=h'^2.\left(\frac{A_2}{12}+\frac{A_1.A_2}{4.A}\right).$$

Cor. 2.–Let y_a be the distance of the compressed, or lower side, from the neutral axis.

Let y_b be the distance of the stretched, or upper side, from the neutral axis.

Let f_a be the crushing unit stress, f_b the tensile unit stress.

From the preceding, $y_a=\frac{h'}{2}.\frac{2A_1+A_2}{A}$; but $h'=y_a+y_b$;

$$\therefore y_a=\frac{y_a+y_b}{2}.\frac{2A_1+A_2}{A_1+A_2}, \text{ and } A_1=A_2.\frac{y_a-y_b}{2.y_b}=A_2.\frac{f_a-f_b}{2.f_b}$$

Hence, I becomes$=\frac{h'^2}{6}.A_2.\frac{2f_a-f_b}{f_a+f_b}$, and $\frac{f_a+f_b}{f_a}=\frac{y_a+y_b}{y_a}=\frac{h'}{y_a}$

$$\therefore I=\frac{h'}{6}.A_2.\frac{y_a}{f_a}.(2.f_a-f_b)$$

(9.)–*To design a girder l-ft. in length to carry a uniformly distributed load of w-lbs. per lineal ft.*

A. To design the flanges.

Assume that the flange areas of any vertical section are the same, and consider a section at a distance x from the centre.

Let h be the effective depth of the section, and t the thickness of either flange.

Let the flanges be of the same uniform width, b, throughout.

$\therefore$ from the preceding,

$$\frac{w}{2}.\left(\frac{l^2}{4}+x^2\right)=M=f.a.h=f.b.t.h$$

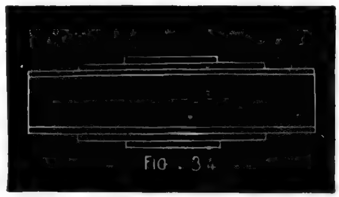
$$\text{and } t.h=\frac{w}{2.f.b}.\left(\frac{l^2}{4}-x^2\right)$$

Now either t or h may vary, and therefore there are two solutions.

Solution 1.–Suppose t to be constant, then h is proportional to the ordinates of a parabola of which the equation is, $h = \frac{w}{2.f.b.t} \cdot \left(\frac{l^2}{4} - x^2\right)$.

When $x = 0$, *i.e.*, at the centre, h is a maximum and equal to $\frac{w.l^2}{8.f.b.t} = d$, suppose. Thus the upper flange of the girder must be parabolic in form, and its central depth $d = \frac{w.l^2}{8.f.b.t}$.

Also, the *volume* of the two flanges is approximately $2.b.t.l = \frac{w.l^3}{4.f.d}$



Solution 2.–Suppose the depth of the girder to be the same throughout, and equal to d, then t is proportional to the ordinates of a parabola of which the equation is $t = \frac{w}{2.f.b.d} \cdot \left(\frac{l^2}{4} - x^2\right)$. When $x = 0$, *i.e.*, at the centre, the thickness of a flange is a maximum and equal to $\frac{w.l^2}{8.f.b.d} = T$, suppose. Thus, the girder is of the form shewn in the Fig., the flanges being built up with plates.

Again, the central sectional area of a flange $= b.T$, and the total theoretic volume of both flanges $= 2.\frac{2}{3} \cdot b.T.l = \frac{w.l^3}{6.f.d}$.

Hence, *theoretically*, the flange volume in the latter case is $\frac{2}{3}$-rds of that in the former, shewing the advantage, in point of *economy*, of girders with horizontal flanges.

At any section, $\frac{E}{R} = \frac{f_y}{y}$, and $\therefore R \propto y$, as E and f_y are both constant. But y is proportional to h, the depth, $\therefore R \propto h$, and the curvature diminishes as h increases, so that a girder with horizontal flanges is superior in point of *stiffness*, to one of the parabolic form.

If great flexibility is required, as in certain dynamometers, the parabolic form of girder is of course the best

Practical considerations may tend to modify the above conclusions, but it must be carefully remembered that the parabolic form of girder

and the girder with horizontal flanges, both obey the laws of *uniform strength.*

B.–To design the web.

The web has to transmit the *Shearing Force* only, and, theoretically should contain no more material than is absolutely necessary for this purpose.

Consider a vertical section at a distance x from the centre, h and t' being respectively, the depth and thickness of the web.

Let f_s be the safe shearing unit stress of the material.

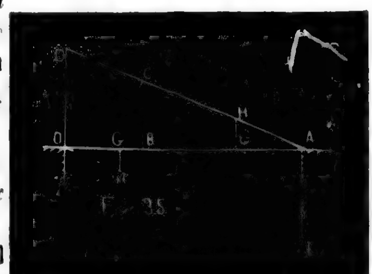
$$\therefore\ h.t' = \text{sectional area of web} = \frac{\textit{Shearing Force}}{f_s} = \frac{w.x}{f_s}.$$

Thus, the sectional area is independent of the depth, and the total volume of the web $\left(=\frac{w.l^2}{4.f_s}\right)$ is the same for the parabolic as for the rectilinear form.

The thickness of the web at the given section is $t'=\frac{w.x}{h.f_s}$, but this is often too small to be of any practical use. Experience indicates that the minimum thickness of a plate which has to stand ordinary wear and tear is about $\frac{1}{4}$ or $\frac{5}{16}$-in., while, if subjected to saline influence, its thickness should be $\frac{3}{8}$ or $\frac{1}{2}$-inch. Hence, the web of a parabolic girder would be much lighter than the web of a girder with horizontal flanges, but this advantage by no means establishes the superiority of the former.

In the case of riveted girders with plate webs of medium size, all practical requirements are effectively met by specifying that the shearing stress is not to exceed *one-half* of the flange tensile stress, and that stiffeners are to be introduced at intervals not exceeding *twice* the depth of the girder when the thickness of the web is less than *one-eightieth* of the depth.

(10).–*To discuss the effect of a rolling load.–Case 1.*–Let a single weight W travel from left to right over a girder OA of length l, resting upon two supports at O and A.



The reaction R_1 at O, when W is at B distant x from O, is $W.\frac{l-x}{l}$ and is the *Shearing Force* for all points between O and B; it is nil or

W according as the weight is at A or O. Upon the vertical through O take OD, equal or proportional to W; join DA. The shearing force at any point of the beam between O and the weight, as the latter travels from A towards O, is represented by the vertical distance between that point and the line AD.

Also, the shearing force at any point between B and A is $R_1 - W = - W.\frac{x}{l}$, and is equal or proportional to the vertical distance between that point and the line OE where AE is equal to OD.

Again, the *Bending Moment* at B, when W is at B, is $W.\frac{l-x}{l}.x$; it is nil at O and at A; it is a maximum and $=\frac{W.l}{4}$ at the middle point D. The bending moment at any point of the beam when the weight is at that point is represented by the vertical distance between the point and the parabola OEA, having its axis vertical and its vertex at E, where DE is equal or proportional to $\frac{W.l}{4}$.

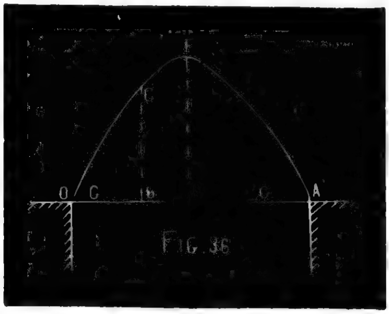


Fig. 36

Note.–The shearing and bending actions are symmetrical on both sides of the centre, and it is therefore sufficient to deal with one-half of the girder only.

Cor. 1.–The shearing force and bending moment at any point are maxima at the instant the weight passes that point.

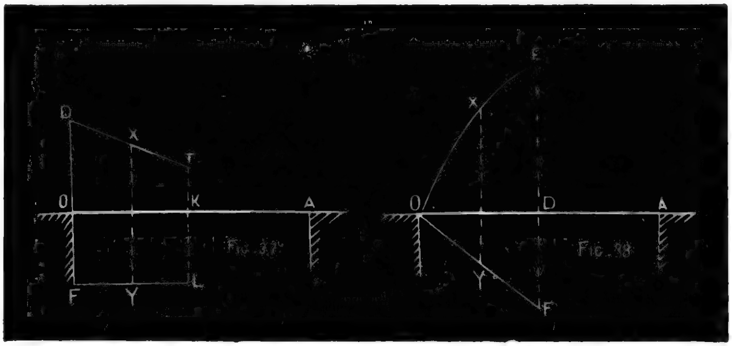
For example, the shearing force at B for the segment OB, when the weight is at B, is equal or proportional to BC, (Fig. 35), which is evidently greater than GH, representing the shearing force at B, when the weight is at any other point G.

Again, the bending moment at B, (Fig. 36) when W is at B, is $W.\frac{l-x}{l}.x$ If W is at any other point G distant a from O, the bending moment at B is $W.a.\frac{l-x}{l}$ or $W.x.\frac{l-a}{l}$, according as $a <$ or $> x$, and in either case is greatest when $a = x$, i.e., when the weight is at B.

Cor. 2.–In addition to the rolling load, let the girder carry a permanent weight W' at the centre.

Consider one-half of the girder only, and, for convenience, trace the shearing force and bending moment diagrams for W' *below OA*.

The compound diagram for maximum shearing forces is *DTLFD*, (Fig. 37), where KT is equal or proportional to $\frac{W}{2}$, and $KL = OF$ is equal or proportional to $\frac{W'}{2}$.



The maximum shearing force at a point distant x from the centre is represented by $XY = \frac{W}{l}\cdot\left(\frac{l}{2}+x\right) + \frac{W'}{2}$.

Again, the compound diagram for maximum bending moments is *OEFO*, (Fig. 38), where DF is equal or proportional to $\frac{W'.l}{4}$, and OF is a straight line.

The maximum bending moment at a point distant x from the centre is represented by

$$XY = \frac{W}{l}\cdot\left(\frac{l^2}{4} - x^2\right) + \frac{W'}{2}\cdot\left(\frac{l}{2} - x\right).$$

Cor. 3.–*Theoretically*, the total volume of material required in the web of the girder in Cor. 2, is equal or proportional to

$$\frac{2.\ Area\,DTLF}{f_s} = \frac{3}{4}\cdot\frac{W.l}{f_s} + \frac{1}{2}\cdot\frac{W'.l}{f_s}.$$

So, if d be the effective depth of the girder, and f the unit-stress in one of the flanges, the total volume of metal in that flange is equal or proportional to

$$\frac{2.\ Area\,OEFO}{f.d} = \frac{2}{3}\cdot\frac{W.l^2}{4.f.d} + \frac{W'.l^2}{8.f.d} = \frac{1}{6}\cdot\frac{W.l^2}{f.d} + \frac{1}{8}\cdot\frac{W'.l^2}{f.d}.$$

Case II.–Let a train weighing w per unit of length travel over the girder from right to left, and let the total length of the train be not less than that of the girder.

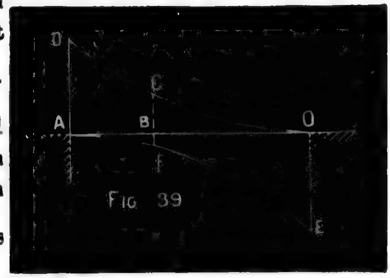


FIG 39

The reaction at A, Fig. 39, when the front of the train is at B distant x from O, is $\frac{w.x^2}{2.l}$, and is the shearing force for all points between A and B. Upon the verticals through A and O take AD and OE each equal or proportional to $\frac{w.l}{2}$. Thus between A and B, the shearing force at any point is represented by the vertical distance between that point and a parabola having its axis vertical and its vertex at O.

After the end of the train has passed O, the shearing force at any point of the uncovered portion of the girder is evidently represented by the vertical distance between that point and the parabola AFE, having its axis vertical and its vertex at A.

Again, as the train moves from O towards B, the reaction at A, and consequently the bending moment at B, continually increase. On passing B, the reaction at A still increases, and the bending moment at B when the train covers a length a of the girder, is

$$\frac{w.a^2}{2l}.(l-x) - \frac{w}{2}.(a-x)^2 = \frac{w.x}{2.l}.a.(2.l-a) - \frac{w.x^2}{2}.$$

This expression is evidently a maximum, when $a.(2.l-a)$ is a maximum i.e., when $a=l$. Hence, the bending moment, and therefore the flange stresses, at any point are greatest, when the moving load covers the whole girder.

Cor. 1.–The shearing force at any point B is a maximum when the train covers the longest segment OB.

This is evidently the case until the train arrives at B, for the reaction at A, and therefore the shearing force at B, will continually increase up to this point. When the train passes B and covers a length a $(>x)$ of the girder, the shearing force at B is $\frac{w.a^2}{2.l} - w.(a-x)$.

But this is $<\frac{w.x^2}{2.l}$, the Shearing Force at B when OB is covered,

$$\text{if } \frac{a^2-x^2}{2.l} < a-x$$

i.e., if $\frac{a+x}{2.l} < 1$, which is evidently the case.

Cor. 2.–In designing the flanges of a girder, the rolling load is supposed to cover the whole girder, and may be treated as a uniformly distributed load.

Cor. 3.–In addition to the rolling load, let the girder carry a uniformly distributed load of w' per unit of length.

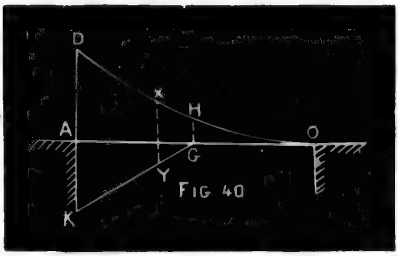


FIG 40

As before, consider one-half of the girder only. Trace the shearing force diagram for the permanent load below OA. The compound diagram is $DHGK$, where GH and AK are equal or proportional to $\frac{w.l}{8}$ and $\frac{w'.l}{2}$, respectively.

The maximum shearing force at a point distant x from the centre is represented by XY, and is equal to $\frac{w}{2.l}\cdot\left(\frac{l}{2}+x\right)^2 + w'.x$

Again, the maximum flange-stresses are obtained by assuming the total load upon the girder to be $w + w'$ per unit of length.

Cor. 4.–*Theoretically*, the total volume of material required in the web of the girder in Cor. 3 is represented by $\frac{2.\,area\ DHGK}{f_s}$

$$= \frac{2.\{area\ DOA - area\ HOG + area\ AGK\}}{f_s} = \frac{7}{24}\cdot\frac{w.l^2}{f_s} + \frac{1}{4}\cdot\frac{w'.l^2}{f_s}.$$

Also, if d is the effective depth of the girder, and f the unit stress for both flanges, the total theoretical volume of the flanges

$$= 2.\frac{2}{3}\cdot\frac{(w+w').\frac{l^2}{8}}{f.d}.l = \frac{1}{6}\cdot\frac{w+w'}{f.d}.l^3$$

In practice, in order to make allowance for stiffeners, covers, rivets, packing, &c., these volumes may be increased by certain per-centages, say m and n respectively. Thus, the actual volume of the web

$= \frac{l^2}{f_s}\cdot\left(\frac{7}{24}.w + \frac{1}{4}.w'\right)(1+m)$, and the actual volume of the flanges

$= \frac{l^3}{6}\cdot\frac{w+w'}{f.d}.(1+n)$; the total volume of the girder

$$= \frac{l^2}{f_s}\cdot\left(\frac{7}{24}.w + \frac{1}{4}.w'\right).(1+m) + \frac{l^3}{6}\cdot\frac{w+w'}{f.d}.(1+n)$$

If l is the length in feet, the total volume in cubic feet is,

$$\frac{1}{144}\cdot\left\{\frac{l^2}{f_1}\cdot\left(\frac{7}{24}.w+\frac{1}{4}.w'\right).(1+m)+\frac{l^3}{6}\cdot\frac{w+w'}{f.d}.(1+n)\right\}.$$

Ex.–The two main girders of a single track bridge are 80-ft. in the clear and 10-ft. deep. The dead load upon the bridge is 2500-lbs. per lineal ft. If the bridge is traversed by a uniformly distributed live load of 3000–lbs. per lineal ft., determine the maximum bending moment and shearing force at a point of the girder distant 10-ft. from one end.

The bending moment at any point is a maximum when the train covers the whole of the bridge, in which case the total distributed load is 5500-lbs. per lineal ft., of which each girder carries one-half.

Thus, the reaction at each support, $=\frac{1}{2}.80.\frac{5500}{2}=110{,}000$–lbs., and the *Bending Moment* at the given point $=110{,}000\times10-10.2750.5$ $=962{,}500$-ft. lbs.

The *Shearing Force* at the given point due to the *dead* load $=110{,}000-10.2750=82{,}500$-lbs.

The *Shearing Force* due to the live load is a maximum when the live load covers the 70-ft. segment, and its value is then

$$\frac{1500.70^2}{2.80}=45{,}937\tfrac{1}{2}\text{-lbs.}$$

Hence, the total maximum shearing force $=82{,}500+45{,}937\frac{1}{2}$ $=128{,}437\frac{1}{2}$-lbs.

Again, if the girder is of wrought iron, and if the safe shearing and flange stresses per sq.-in. of the metal are 7500-lbs. and 10,000-lbs. respectively, the weight of the girder is, according to Cor. 4 of the above,

$$\frac{480}{144}\cdot\left\{\frac{6400}{7500}\cdot\left(\frac{7}{24}.1500+\frac{1}{4}.1250\right)(1+m)+\frac{512{,}000}{6}\cdot\frac{2750}{10{,}000.10}.(1+n)\right\}$$

$$=\left\{\frac{6400}{3}(1+m)+\frac{70{,}400}{9}(1+n)\right\}\text{lbs.}$$

Suppose, for example, that m is about $12\frac{1}{2}$ p.c., and n 25 p.c.

$\therefore$ the weight of the girder$=\frac{6400}{3}\cdot\left(1+\frac{1}{8}\right)+\frac{70{,}400}{9}\cdot\frac{5}{4}=12177\frac{7}{9}$-lbs.

(11).–*To make allowance for the weight of a Beam.*–A beam is sometimes of such length that its weight becomes of importance as compared with the load it has to carry. In such a case the dimensions of the beam may be determined in the following manner:—

Assume that the beam is of uniform section and of constant depth.

Also assume that the working and dead loads are uniformly distributed.

Let E_1 be the working load, and f_1 its factor of safety.

Let b_1 be the breadth of any part of a beam of which the breaking load is $f_1.E_1$.

Let $f_2.W_1$ be the weight of this beam, f_2 being the factor of safety for a steady load like the weight of a beam.

$\therefore f_1.E_1 - f_2.W_1$ is the *net* load, and $\dfrac{f_1.E_1}{f_1.E_1 - f_2.W_1} = \dfrac{\text{the } gross \text{ load}}{\text{the } net \text{ load}}$

But the *weight* of the beam, the *working* load, and consequently the *net* load, all vary directly with a breadth of the beam, so that if b is the breadth, corresponding to b_1, of a beam of which the *net* load is to be $f_1.E_1$, $\therefore \dfrac{f_1.E_1}{f_1.E_1 - f_2.W_1} = \dfrac{b}{b_1}$.

Hence, if $f_1.E$, and $f_2.W$, are respectively, the breaking load and weight of the new beam,

$$\therefore \frac{f_2.W}{f_2.W_1} = \frac{f_1.E}{f_1.E_1} = \frac{b}{b_1} = \frac{f_1.E_1}{f_1.E_1 - f_2.W_1}.$$

Also, the net load of the new beam $= f_1.E - f_2.W$

$$= \frac{f_1^2.E_1^2}{f_1.E_1 - f_2.W_1} - f_2.W_1.\frac{f_1.E_1}{f_1.E_1 - f_2.W_1} = f_1.E_1.$$

$$\text{and } \therefore f_1.E = f_2.W + f_1.E_1.$$

Ex.–Apply the above results to a cast-iron girder of rectangular section, resting upon two supports 30-ft. apart. The girder is 12-ins. deep, and carries a uniformly distributed load of 30,000-lbs.

$$E_1 = 30{,}000\text{-lbs.} \quad \text{Let } f_1 = 4, \therefore f_1.E_1 = 120{,}000\text{-lbs.}$$

b_1 is given by $\dfrac{120{,}000}{2} = C.\dfrac{b_1.d^2}{l}$, where C is 30,000, d is 12 ins., and l is 360-ins.

$$\therefore 60{,}000 = 30{,}000.\frac{b_1.144}{360}, \text{ and } b_1 = 5\text{-ins.}$$

Hence, $f_2.W_1 = 5.12.360.\dfrac{450}{1728} = 5625\text{-lbs.}$, and $f_1.E_1 - f_2.W_1 = 114{,}375\text{-lbs.}$

$$\therefore \frac{b}{5} = \frac{120{,}000}{114{,}375}, \text{ and } b \text{ is } 5\tfrac{1}{4}\text{-ins. nearly.}$$

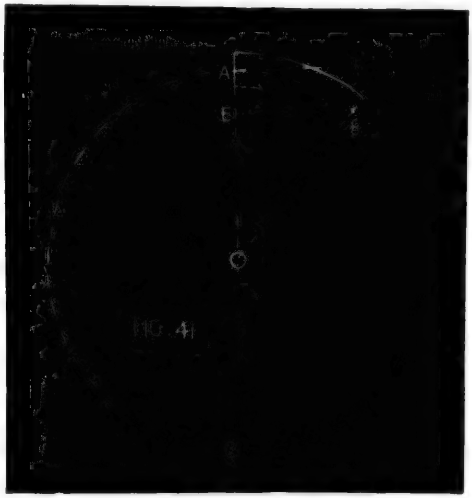
$$\text{Also, } f_1.E = \frac{b}{b_1}.f_1.E_1 = \frac{5\frac{1}{4}}{5}.120{,}000 = 126{,}000\text{-lbs.}$$

$$\text{and } f_2.W = \frac{b}{b_1}.f_2.W_1 = \frac{5\frac{1}{4}}{5}.5625 = 5906.25\text{-lbs.}$$

(12) –*On the Deflection of Girders.*–The principles of economy and strength require a girder to be designed in such a manner that every part of it is duly proportioned to the greatest stress to which it may be subjected. When such a girder is acted upon by external forces, it is

uniformly strained throughout,, and in bending, the neutral axis must necessarily assume the form of an arc of a circle. It might be supposed that the curve of deflection is dependent upon the character of the web, but careful experiments indicate that the web exercises very little, if any, influence, so long as the flange unit stresses are unaltered in amount.

Ex. 1.–A semi-girder is bent under the action of external forces, and its neutral axis, AB, forms an arc of the circle ABC.



Draw BF vertically to meet the horizontal line through A in F.

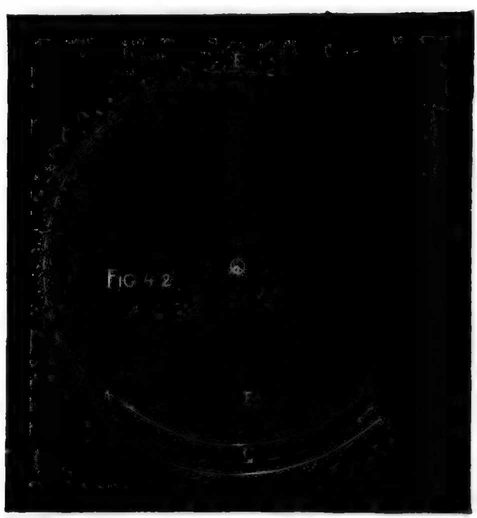
Draw BE horizontally to meet the vertical diameter AC in E.

Let $BF = D$; D is the maximum deflection of the girder.

Let the radius of the circle $= R$.

$$\therefore BE^2 = AE.EC = AE.(AC - AE) = BF.(AC - BF) = D.(2R - D).$$

Now, since the deflection is small, BE is approximately equal in length to $AB (= l)$, the neutral axis, and D^2 may be neglected without serious error,



$\therefore l^2 = 2.R.D$, and

$$D = \frac{l^2}{2R}.$$

Cor. 1.–The deflection Δ at a point distant x from A is evidently $\dfrac{x^2}{2.R}$

Ex. 2.–A girder resting upon two supports is bent under the action

of external forces, and its neutral axis ACB forms an arc of the circle ABE.

Let the vertical diameter EC meet the horizontal line AB in F.

Let $CF=D$; D is the maximum deflection of the girder.

Let the radius of the circle $=R$.

$$\therefore AF^2=CF.FE=CF.(CE-CF)=D.(2R-D)$$

Now, since the deflection is small, the line AB is approximately equal in length to ACB $(=l)$, the neutral axis, and D^2 may be neglected without serious error.

$$\therefore \left(\frac{l}{2}\right)^2=2\,R.D, \text{ and } D=\frac{l^2}{8.R}.$$

Cor.–The deflection Δ at a point distant x from CF, is given by,

$$x^2=(D-\Delta)\ (2\,R-\overline{D-\Delta})=2\,R.(D-\Delta), \text{ approximately.}$$

$$\therefore \Delta=D-\frac{x^2}{2\,R}.$$

Ex. 3.–Let s_1,f_1,d_1, and s_2,f_2,d_2, respectively, be the length, unit stress, and distance from the neutral axis, of the stretched and compressed outside fibres in Examples (1) and (2).

Let $d_1+d_2=d=$ the total depth of the girder.

$\therefore$ from similar figures,

$$\frac{s_1}{l}=\frac{R+d_1}{R}, \text{ and } \frac{s_2}{l}=\frac{R-d_2}{R}, \therefore \frac{s_1-s_2}{l}=\frac{d_1+d_2}{R}=\frac{d}{R}.$$

Also, $\frac{f_1}{E}=\frac{s_1-l}{l}=\frac{d_1}{R}$, and $\frac{f_2}{E}=\frac{l-s_2}{l}=\frac{d_2}{R}$

$$\therefore \frac{f_1+f_2}{E}=\frac{s_1-s_2}{l}=\frac{d_1+d_2}{R}=\frac{d}{R}.$$

Note.–A measure of the stiffness of the girder is $\frac{W}{D}$, W being the total load upon the girder, and D the maximum deflection due to such load.

Ex. 4.–The deflection D of a beam loaded with a weight $W \propto \frac{l^2}{R}$.

The maximum bending moment M of such a beam $\propto W.l$

But $M=\frac{E}{R}.I$; $\therefore D \propto \frac{M.l^2}{E.I} \propto \frac{W.l^3}{E.I}$

Let b and d, respectively, be the breadth and depth of a rectangular beam; $\therefore I=\frac{b.d^3}{12}$.

Hence, for rectangular beams of the same material,

$$D \propto \frac{W.l^3}{b.d^3}=m.\frac{W.l^3}{b.d^3} \qquad (1)$$

m being a coefficient to be determined by experiment.

[At the end of the chapter is a table of the values of m as given by the most reliable authorities].

Instead of (1), Prof. Norton has proposed the following for timber beams :—

$$D = \frac{W.l^3}{4\,E\,b.d^3} + m.\frac{W.l}{b.d},$$

m being .0000094, and E=1,427,965-lbs.

Assuming that (1) gives the deflection of a beam resting on two supports and loaded with the weight W at the centre,

The deflection of the beam with the same load uniformly distributed $= \frac{5}{8}.D$

" " firmly fixed at both ends and loaded in the centre $= \frac{1}{4}.D$

" " " " " loaded uniformly $= \frac{1}{8}.D$

" " " " one end and loaded at the other $= 16.D$

" " partially fixed at one end and loaded at the other $= 24.D$ *

" " firmly " " and loaded uniformly $= 12.D$

It is generally assumed that the greatest deflection of a *timber* beam should not exceed $\frac{1}{360}$th its length.

According to Rankine, the safe deflection of a beam resting upon two supports should lie between $\frac{\text{span}}{600}$ and $\frac{\text{span}}{1200}$ for the *working* load, and between $\frac{\text{span}}{200}$ and $\frac{\text{span}}{600}$ for the *proof* load.

Example–A timber girder of rectangular section is 12-ins. deep, 6-ins. wide, and rests upon two supports 20-ft. apart. Determine the uniformly distributed load which will cause a deflection of *one inch* at the centre. (E=1,200,000).

Here, l=240-ins, and D=1-in., $\therefore 1 = \frac{(240)^2}{8.R}$, and R=7200-ins.

Again, $\frac{E}{R}.I = M = \frac{W.l}{8}$, where W is the required load.

$\therefore \frac{1{,}200{,}000}{7200}.\frac{6.12^3}{12} = \frac{W.240}{8}$, and W=48,000-lbs.

* Trautwine.

Example.—A trellis girder resting upon two supports 120-ft. apart is 15-ft. deep, and is strained in such a manner that the flange tensile and compressive unit-stresses are 10,000-lbs per sq. in., and 8,000-lbs per sq. in., respectively. Determine the central deflection, and the difference in length between the extreme fibres. (E=30,000,000).

From example (3), $\dfrac{10,000+8,000}{30,000,000}=\dfrac{15}{R}=\dfrac{s_1-s_2}{120}=\dfrac{3}{5,000}$.

$\therefore R$=25,000-ft., and $s_1-s_2=\dfrac{18}{250}$-ft.=.864-ins.=required difference in length.

Also, $D=\dfrac{(120)^2}{8.25,000}$= 864-ins.=central deflection.

(13).—*Beam acted upon by forces oblique to its direction, but lying in a plane of symmetry.*—In discussing the equilibrium of such a beam the forces may be resolved into components parallel and perpendicular to the beam, and their respective effects superposed.

Let AB be the beam, $P_1,P_2,P_3,\ldots\ldots$ the forces, and $a_1,a_2,a_3,\ldots\ldots$ their respective inclinations to the neutral axis.

Divide the beam into any two segments by an imaginary plane MN perpendicular to the beam, and consider the segment AMN.

It is kept in equilibrium by the external forces on the left of MN and by the elastic reaction of the segment BMN upon the segment AMN at the plane MN.

The resultant force along the beam, is the algebraic sum of the components in that direction of $P_1,P_2,P_3\ldots\ldots = P_1.cos.a_1 + P_2.cos\ a_2 + \ldots\ldots = \Sigma(P.cos\ a)$.

It may be assumed that this force acts along the neutral axis, and is uniformly distributed over the section MN.

Thus, if A is the area of the section, $\dfrac{\Sigma(P.\ cos\ a)}{A}$ is the *intensity* of stress due to this force.

Again, the components of P_1, P_2, P_3......perpendicular to the beam, are equivalent to a *single force* and a *couple* at MN.

The single force at MN is the *Shearing Force*, is perpendicular to the beam, and is the algebraic sum of $P_1.\sin a_1$, $P_2.\sin a_2$, ...

$$=P_1.\sin a_1 + P_2.\sin a_2 + - - = \Sigma(P.\sin a).$$

This force develops a mean *tangential* unit stress of $\dfrac{\Sigma(P.\sin a)}{A}$ in MN, and deforms the beam, but so slightly as to be of little account.

The moment of the couple is the algebraic sum of the moments with respect to MN of $P_1.\sin a_1$, $P_2.\sin a_2$,......$=P_1.\sin a_2.p_1 + P_2.\sin a_2.p_2$ $+$......$=\Sigma(P.p.\sin a)$., p_1, p_2,.. ..being respectively the distances of the points of application of P_1, P_2......from MN.

Now, $\Sigma(P.p.\sin a)$ is the resultant moment of *all* the external forces on the left of MN, for the resultant moment of the components along the beam is evidently *nil*.

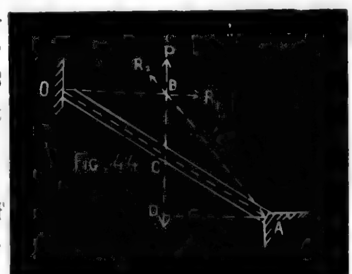
$$\therefore \Sigma(P.p.\sin a) = M = \frac{E}{R}.I = \frac{f_y}{y}.I.$$

$\therefore f_y = \dfrac{y}{I}.\Sigma(P.p.\sin a)$, is the unit stress in the material of the beam at a distance y from the neutral axis due to the bending action at MN of the external forces on the segment AMN.

Hence, also, the *total* unit stress in the material in the plane MN at a distance y from the neutral axis is,

$$\pm\frac{\Sigma(P.\cos a)}{A} \pm f_y = \pm\frac{\Sigma(P.\cos a)}{A} \pm \frac{y}{I}.\Sigma(P.p.\sin a) = f'_y,$$

the signs depending upon the *kind* of stress.



Ex.–The inclined beam OA, carrying a uniformly distributed load of w per unit of length, is supported at B and rests against a vertical surface at O.

The resultant weight, $w.l.$, is vertical, and acts through the centre C of OA; the reaction R_1 at O is horizontal.

Let the directions of $w.l$ and R_1 meet in B. For equilibrium, the reaction R_2 at A, must also pass through B.

Let the vertical through C meet the horizontal through A in D.

The triangle ABD is a triangle of forces for the three forces which meet at B.

$$\therefore \frac{R_1}{w.l} = \frac{AD}{BD} = \frac{AD}{2.DC} = \frac{1}{2}.\ cot\ a,\ a \text{ being the angle } OAD.$$

$$\therefore R_1 = \frac{w.l}{2}.\ cot\ a.$$

Consider a section MN perpendicular to the beam at a distance x from O.

The only forces on the left of MN are R_1 and the weight upon OM. This last is $w.x$, and its resultant acts at the centre of OM, *i.e.*, at a distance $\frac{x}{2}$ from MN.

The component of R_1 along the beam $= R_1.cos\ a = \frac{w.l}{2}.\frac{cos^2 a}{sin\ a}$

" " R_1 perpendicular to the beam $= R_1.sin\ a = \frac{w.l}{2}\cdot cos\ a$

" " $w.x$ along the beam $= w.x\ sin\ a.$

" " " perpendicular to the beam $= w.x\ cos\ a.$

Hence,

The total compression is $NM = \frac{w.l}{2}.\frac{cos^2 a}{sin\ a} + w.x.sin\ a = C_x.$

The shearing force at $MN = \frac{w.l}{2}\cdot cos\ a - w.x\ cos\ a = S_x.$

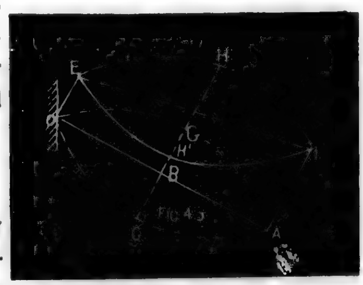
The bending moment at $MN = \frac{w.l}{2}\cdot cos\ a.x - w.x.cos\ a.\frac{x}{2} = M_x.$

$$\text{and } f'_y = \frac{C_x}{A} \pm \frac{y}{I}\cdot M_x.$$

These expressions may be interpreted graphically as already described, C_x, S_x, being represented by the ordinates of straight lines, and M_x, f_y, by the ordinates of parabolas.

f_y, for example, consists of two parts which may be treated independently. Draw OE and AF perpendicular to OH, and respectively equal or proportional to

$\frac{w.l}{2.A}.\frac{cos^2 a}{sin\ a}$ and $\frac{w.l}{2.A}.\frac{cos^2 a}{sin\ a} + \frac{w.l}{A}.sin\ a.$



Join EF. The unit stress at any point of the beam due to direct compression is represented by the ordinate (drawn parallel to OE or AF) from that point to EF.

Upon the line GG' drawn through the middle point B perpendicular to OA, take $BG=BG'$, equal or proportional to $\frac{y}{I}.\frac{w.l^2}{8}.\cos a$. According as the stress due to the bending action at any point of the beam is compressive or tensile, it is represented by the ordinate (drawn parallel to OE and AF) from that point to the parabola OGA or $OG'A$; G and G', respectively, being the vertices, and GG' a common axis.

By superposing these results. the parabolas EHF, $EH'F$, are obtained, the ordinates of which are respectively proportional to the values of f_y for the compressed and stretched parts of the beam, *i.e.*, for the parts above and below the neutral surface.

(14).–*Similar Girders.*–Two girders are said to be *similar*, when the linear dimensions of the one bear the same constant proportion to the corresponding linear dimensions of the other.

Thus, if β, β', δ, δ', λ, λ', are corresponding breadths, depths and lengths, of two similar girders, $\therefore \frac{\beta}{\beta'}=\frac{\delta}{\delta'}=\frac{\lambda}{\lambda'}=$ a constant $=\mu$, suppose.

(15).–*To deduce the principal properties of similar girders.*

(a).–The weight of a girder is proportional to the product of an area and length, *i.e.*, to the cube of a linear dimension.

$\therefore$ *the weights of similar girders vary directly as the cubes of their linear dimensions.* Hence, too, *the unit stresses must vary directly as their linear dimensions.*

(b).–The *Breaking Weight* of a girder is calculated from a formula of the form $W=S.\frac{a.d}{l}$, a being an area, d a depth, and l a length.

Now $\frac{d}{l}$ is constant for similar girders, so that W is proportional to a, *i.e.*, to the square of a linear dimension.

$\therefore$ *The Breaking Weights of similar girders vary directly as the squares of their linear dimensions.*

Ex.–A girder resting upon two supports 80-ft. apart is 10-ft. deep, and weighs 6 tons. Determine the length and depth of a similar girder weighing 48 tons, $\left(\frac{length}{80}\right)^3=\left(\frac{depth}{10}\right)^3=\frac{48}{6}=8$, $\therefore$ the length $=160$ ft., and the depth $=20$-ft. Also the unit stresses are in the ratio of 10 to 20, and the breaking weights in the ratio of 10^2 to 20^2.

(16).—*To discuss the relations between the corresponding sectional areas, moments of inertia, weights, bending moments, &c.,......of two girders which have the same sectional form and are thus related:—*

The forces upon the one being P_1, P_2, P_3,......with abscissæ x_1, x_2, x_3...
those upon the other are $n.P_1, n.P_2, n.P_3$,...... " $p.x_1, p.x_2, p.x_3$

The spans and corresponding lengths are in the constant ratio p.
Corresponding sectional breadths " " " q.
Corresponding sectional depths " " " r.

Let A,A′, be corresponding sectional areas,
" I, I', " " moments of inertia,
" Q, Q', " " weights,
" S, S', " " shearing forces,
" M, M', " " bending moments,
" f, f', " " flange unit stresses,
" s, s', " " web unit stresses,
" R, R', " " radii of curvature,
" Δ, Δ', " " deflections,
" W, W', " " breaking weights.

$\therefore$ (α).—$A \propto$ product of a breadth and depth; $\therefore A' = A.q.r.$

(β).—I " breadth and the cube of a depth; $\therefore I' = I.q.r^3$

(γ).—Q " length, breadth and depth: $\therefore Q' = Q.n = Q.p.q.r.\rho$,

ρ being the ratio of the specific weights of the materials of the girders. If the materials are the same, $\therefore \rho = 1$, and $n = p.q.r.$

(δ).—$S' = S.n. = S.p.q.r.\rho$, for from (γ) $n = p.q.r.\rho$.

(ϵ).—M is the product of a force and a length; $\therefore M' = M.n.p. = M.p^2.q.r.\rho$.

(ζ).—$f = \dfrac{c.M}{I}$, and $f' = \dfrac{c'.M'}{I'}$, $\therefore \dfrac{f'}{f} = \dfrac{c'}{c}.\dfrac{M'}{M}.\dfrac{I}{I'} = r.n.p.\dfrac{1}{q.r^3} = \dfrac{p^2}{r}.\rho$.

(η).—$s = \dfrac{S}{A}$, and $s' = \dfrac{S'}{A'}$, $\therefore \dfrac{s'}{s} = \dfrac{S'}{S}.\dfrac{A}{A'} = \dfrac{n}{q.r} = p.\rho$.

(θ).—$\dfrac{E}{R} = \dfrac{f}{c}$, and $\dfrac{E'}{R'} = \dfrac{f'}{c'}$, E and E' being the co-efficients of elasticity of the respective girders.

$$\therefore \frac{R'}{R} = \frac{f}{f'}.\frac{c'}{c}.\frac{E'}{E} = \frac{E'}{E}.r.\frac{r}{p^2\rho.} = \frac{E'}{E}.\frac{r^2}{p^2}.\frac{1}{\rho}.$$

(ι).—Δ is proportional to $\dfrac{(a\ length)^2}{(radius\ of\ curvature)}$; $\therefore \dfrac{\Delta'}{\Delta} = p^2.\dfrac{R}{R'} = \dfrac{E}{E'}.\dfrac{p^4}{r^2}.\rho$.

(κ).—W is " " $\dfrac{(the\ product\ of\ an\ area\ and\ depth)}{(a\ length)}$;

$$\therefore \frac{W'}{W} = \frac{q.r.r}{p} = \frac{q.r^2}{p}.$$

Hence, the radius of $A', I', Q', \ldots\ldots$ and may be deri ed from those of $A, I, Q\ldots\ldots$ by means of certain constant multipliers.

Cor. 1.–If the two girders are *similar*, and of the same material,

$$\therefore p=q=r=\mu,\ E=E', \text{ and } \rho=1.$$

Hence,

from (γ), $Q'=Q.\mu^3$; and the weights vary directly as the cubes of the linear dimensions.

" (ϵ), $M'=M.\mu^4$; and the bending moments vary directly as the fourth powers of the linear dimensions.

" (ζ) and (η), $\frac{f'}{f}=\mu=\frac{s'}{s}$; and the flange unit stresses vary directly as the web unit stresses.

" (θ), $\frac{R'}{R}=1$

" (ι), $\frac{\Delta'}{\Delta}=\mu^2$; and the deflections vary directly as the squares of the linear dimensions.

" (κ), $\frac{W'}{W}=\mu^2$, and the breaking weights vary directly as the squares of the linear dimensions.

Cor. 2.–Let the girders be of the same material, of equal length, of equal rectangular sectional areas, and equally loaded.

Let b, b_1, d, d_1, respectively, be the breadth and depth of the girders.

$$\therefore b_1=q.b, \text{ and } d_1=r.d,\ \therefore\ b_1.d_1=q.r.b.d; \text{ but } b_1.d_1=b.d.$$

$$\therefore q.r=1. \quad \text{Also } p=1.$$

$\therefore$ from ζ, $\frac{f'}{f}=\frac{1}{r}=q.$

" θ and i, $\frac{R'}{R}=r^2=\frac{\Delta}{\Delta'}$.

" κ., $\frac{W'}{W}=r=\frac{f}{f'}$.

Thus, if $d_1=b_1\ \therefore\ b_1=d$, and $r=\frac{b}{d}$.

$$\therefore \frac{f'}{f}=\frac{d}{b}, \text{ and } \frac{R'}{R}=\left(\frac{b}{d}\right)^2=\left(\frac{W'}{W}\right)^2.$$

* Table of the values of *m*, the co-efficient of deflection.

Ash	.00030	Maple	.00040
Beech	.00030	Mahogany, Spanish	.00030
Birch	.00030	" Honduras	.00025
Cedar	.00030	Oak, minimum	.00025
Cherry	.00040	" maximum	.00050
Chestnut	.00025	" mean	.00040
" Spanish	.00050	Pine, White	.00025
Elm	.00030	" Pitch	.00030
Fir, Am. Spruce	.00025	Teak	.00030
" Norway	.00025	Walnut	.00025
Larch	.00030	Willow	.00006

* (*The Materials of Engineering.*—THURSTON.)

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† Table of the resistance to transverse shearing, rupture being produced across the axis of the piece.

MATERIAL.	LBS. PER SQ. IN.	MATERIAL.	LBS. PER SQ. IN.
Ash	6,280	Hickory	6,045 to 7,285
Beech	5,223	Locust	7,176
Birch	5,595	Maple	6,355
Cedar, White	1,372 to 1,519	Oak, White	4,425
" C. Am	3,410	" Live	8,480
Cherry	2,945	Pine, White	2,480
Chestnut	1,535	" Yellow	4,340 to 5,735
Dogwood	6,510	Poplar	4,418
Ebony	7,750	Spruce	3,255
Gum	5,890	Walnut, Black	4,725
Hemlock	2,750	Walnut, White	2,830

† TRAUTWINE.

EXAMPLES.

(1).–A uniform rigid bar weighs W-lbs., and is supported by two strings (assumed straight) AC, BD, attached to its ends; find the tensions in the strings and the inclination of the bar when the strings are inclined to the vertical at angles of 60° and 30°, respectively.

(2).–The ends of a string are fastened to points A and B in the same horizontal line, d-ft. apart, and weights W_1, W_2, are suspended from points C D, of the string, dividing it into segments AC, CD, DB, of which the lengths are l_1, l_2, l_3 ft., respectively. Assuming each portion of the string to be straight, find the position of equilibrium.

(3).–Two timber ribs AC, BC, of equal length and 12-ins. deep, are movable in a horizontal plane about hinges at A and B. They are turned so as to touch at C along a plane perpendicular to AB, and are then subjected to a uniformly distributed pressure of 1200-lbs. per lineal ft. If AB is 65-ft., and if the centre of the joint C is 9-ft. from AB, determine the magnitudes and directions of the resultant pressures at C and at the hinges.

(4).–A heavy uniform beam rests against a rough horizontal plane and a smooth vertical wall, the vertical plane through the beam being perpendicular to the ground and wall; determine the *limiting* position of equilibrum, when a weight is suspended from any given point of the beam.

(5).–A heavy uniform beam rests upon two inclined planes which intersect in a horizontal line, the line being perpendicular to the vertical plane through the beam; find the *limiting* positions of equilibrum, and the corresponding pressures upon the planes, when a weight is placed at a given point of the beam.

(6).–Two heavy uniform beams, AB, BC, are jointed at B, and rest with the ends A and C against a rough horizontal plane and a smooth vertical wall, the vertical plane through the beams being perpendicular to the ground and wall; find the relative position of the beams, so that they may undergo equal horizontal pressures.

(7).–A heavy carriage wheel is to be dragged over an obstacle on a horizontal plane by a horizontal force applied to the centre of the wheel; find the magnitude of the force (neglect the friction between the wheel and obstacle). What happens if the line of action of the force does not pass through the centre? What is the effect of the friction between the wheel and obstacle?

(8).–A heavy mast movable in a vertical plane about a hinge at its foot is held in a given position by a rope fastened to the other end; find, *geometrically*, the least pressure on the hinge, and the corresponding direction of the rope.

(9).–Two equal carriage wheels with their centres connected by a rigid bar, are placed upon a rough plane; is the equilibrium of the system best preserved by locking the hind or fore-wheel?

(10).–The upper ends of two uniform heavy beams abut against each other, and their feet rest upon a rough horizontal plane; shew how to cut a plane face from the upper end of one of the beams, so that slipping may be about to ensue at the surface of contact.

(11).–A car of weight, *W* for a 4-ft. $8\frac{1}{2}$-ins. gauge, is 33-ft. long, 6-ft. deep, and its bottom is 2-ft. 6-ins. above the rails; find the additional weight thrown upon the leeward rail when the wind blows upon a side of the car with a pressure of 20-lbs. per sq. ft.

Find the minimum wind pressure that will blow the car over.

(12).–A beam 40-ft. long carries a load of 20,000-lbs.; find the shearing force at 15-ft. from one end and also the maximum bending moment of the beam :—

(*a*).–When the beam is supported at the ends and loaded in the middle.

(*b*).–When the beam is supported at the ends and loaded uniformly.

(*c*).–When the beam is fixed at one end and loaded at the other.

(*d*).–When the beam is fixed at one end and loaded uniformly.

Draw the curves of shearing force and bending moment, and explain the connection between them.

(13).–Discuss the effect produced in each of the cases of Question (12), *first* when a single weight of 2,000-lbs. passes over the beam, *second* when a train weighing 2,000-lbs. per lineal ft. moves across the beam.

(14).–A beam 20-ft. in length rests upon two supports, and carries a weight of 10-tons at 5-ft. from one end; find the maximum bending moment.

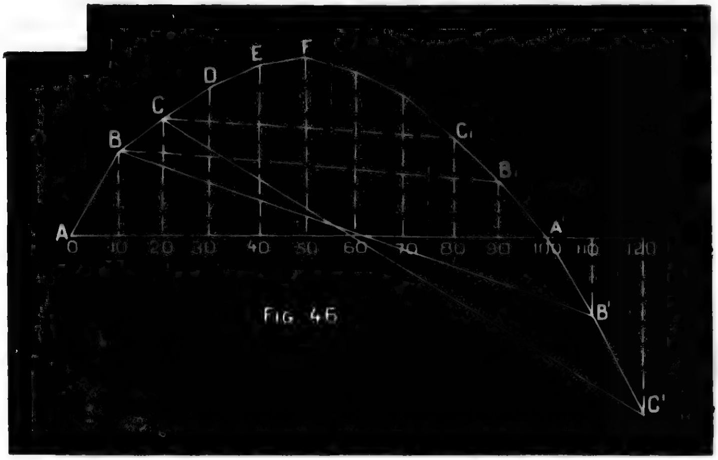
Draw the curves of shearing force and bending moment.

(15).–A double flanged railway girder is 267-ft. in the clear and $22\frac{1}{2}$-ft. deep. Three locomotives weighing 40-tons each rest upon the girder at points 20-ft., 75-ft., and 240-ft. from the left abutment; find the shearing force and bending moment at 180-ft. from the left abutment.

Draw the curves of shearing force and bending moment.

(16).–A girder rests upon two supports at *O* and *A*, and carries an arbitrarily distributed dead load; prove, *graphically*, that the *total* amount of *positive* shearing force is equal to the *total* amount of *negative* shearing force.

Weights are suspended from points 1, 2, 3,...of the girder, dividing it into segments *O*1, 1 2, 2 3,... of which the lengths are $s_1, s_2, s_3, \ldots$ respectively; if $S_1, S_2, S_3, \ldots$ are the corresponding shearing forces, shew that $S_1 . s_1 + S_2 . s_2 + S_3 . s_3 + \ldots = 0$.



(17).–$A\ B\ C\ D\ E\ F\ C_1\ B_1\ A'\ B'\ C'$...is the bending moment curve of a girder resting upon supports 100-ft. apart, and carrying an arbitrarily distributed load concentrated at 10-ft. intervals. If the loads move 10-ft. to the left in the same relative order and distance, prove that the new bending moments along the girder are the vertical ordinates to the line BB', that CC' is the base when the loads have moved 20-ft. to the left, and so on, the lines BB', CC', &c., being inclined to the horizontal at certain angles depending on the loading.

Shew that the above diagram will, for the same loading, give the bending moments for girders of spans of 80-ft, 60-ft., 40-ft., &c., BB_1 being the base for the 80 ft. span, CC_1 for the 60-ft. span, &c.

(18).–A man of weight W ascends a ladder of length l which rests against a smooth wall and the ground, and is inclined to the vertical at an angle a. The ladder has n rounds; find the bending moment at the r-th round from the foot when the man is on the p-th round from the foot.

(19).–A round beam and a square beam are equal in length and equally loaded; find the ratio of the diameter to the side of the square, so that the two beams may be of equal strength.

(20).–Compare the relative strengths of two beams of the same material of which the sections are precisely similar in shape and have areas in the ratio of 1 to 4.

(21).–Compare the relative strengths of a cylindrical beam and the strongest rectangular and square beams that can be cut from it.

Compare the relative strengths of a round and a square beam, the diameter of the round being equal to a side of the square.

(22).–A boiler plate tube, 36-ft. long, 30-ins. inside diameter, weighs 4,200-lbs. and rests upon supports 33-ft. apart; find the stress in the metal at the centre.

What additional weight may be suspended from the centre, consistent with the condition that the stress in the metal is nowhere to exceed 8,000-lbs. per sq. in?

(23).–A beam of yellow pine 14.ins. wide, 15-ins. deep, and resting upon supports 10-ft. 9-ins. apart, was just able to bear a weight of 34-tons at the centre; what weight will a beam of the same material, 3-ft. 9-ins. between the supports and 5-ins. square, bear?

(24).–Determine the breadth of timber joists l-ins. long, d-ins. deep, and x-ins. centre to centre:–(a).–For a wooden platform the gross load being 150-lbs. per sq. ft.; (b).–For a wooden platform with a broken stone or gravel roadway, the gross load being 250-lbs. per sq. ft.; (c).–For a railway platform with a single or double track, W being the heaviest load upon a pair of drivers and k the gauge of the rails.

(The safe stress in the timber $=1{,}000$-lbs. per sq. in.)

(25).–Determine the form of a beam of uniform resistance.

(1).–When the beam rests upon two supports and is uniformly loaded.

(2).–When the beam rests upon two supports and is loaded at the centre.

(3).–When the beam is fixed at one end and uniformly loaded.

(4).–When the beam is fixed at one end and loaded at the other.

(5).–When the beam in cases (1) and (3) carries an additional weight at the centre and end respectively.

Give instances of the practical application of the above.

(26).–A wrought iron stand pipe at Milwaukee, 133-ft. in length, and 30-ins. in diameter (inside), was made in four equal sections. The thickness of the plates in lowest or first section was $\frac{7}{16}$ in.; in the 2nd, $\frac{6}{16}$ in.; in the 3rd, $\frac{5}{16}$ in.; and in the 4th, $\frac{4}{16}$ in., the corresponding section weights being 5,730-lbs., 4,742-lbs., 3,868-lbs., and 3,020-lbs.

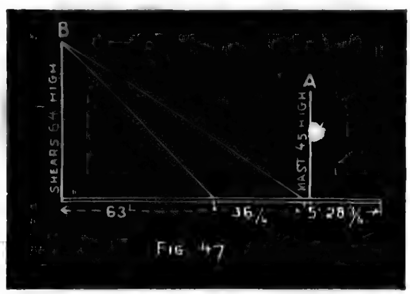


FIG 47

The pipe was riveted together, and raised in one piece as follows:— the pipe was placed with the lower end exactly over its seat, and fixed to a beam of timber turning in bearings. The upper end was lifted 40-ft. by the vertical tackle at A (the inclined ropes being also used), which was then cut away, and the erection completed by the inclined ropes at B.

Assuming these ropes to be equally strained, find the stresses in them when the pipe had been raised through an angle of 60°, and also the resultant re-action at the foot of the pipe. Again, for the given position:–

(1).–Determine the direction and magnitude of the resultant rope stress that will make the resultant reaction at the foot of the pipe a minimum.

(2).–Determine the magnitude of the bending moment, shearing force, and longitudinal compressive stress at the foot of each section.

(3).–Draw the curves of shearing force, bending moment, and compressive stress.

(27).–Compare the strengths of two rectangular beams of equal length, the breadth and depth of one, being respectively equal to the depth and breadth of the other.

(28).–A cast iron beam 4-ins. square rests upon supports 6-ft. apart; determine the breaking weight at the centre. (C=30,000-lbs.)

(29).–The flooring of a corn warehouse is supported upon yellow pine beams 20-ft. in the clear, 8-ins. wide, 10-ins. deep, and spaced 3-ft. centre to centre; find the height to which corn, weighing $48\frac{1}{2}$-lbs. per cube ft. may be heaped upon the floor. (C = 3,000-lbs.)

(30).–A yellow pine beam 14-ins. wide, 15-ins. deep, and resting upon supports 10-ft. 6-ins. apart, broke down under a uniformly distributed load of 60·97-tons; find the coefficient of rupture (C).

(31).–The ribs in Question (3) being of greenheart, find their breadth. (C = 9,000-lbs., and factor of safety = 10).

(32).–Find the breaking weight at the centre of a piece of Canadian elm $2\frac{1}{2}$-ins. wide, $3\frac{1}{2}$-ins. deep, and resting upon supports 3-ft. 9-ins. apart. (C=7,250-lbs.)

(33).–A cast-iron rectangular girder rests upon supports 12-ft. apart and carries a weight of 2000-lbs. at the centre; if the breadth is one-half the depth, find the sectional area of the girder, so that the inch stress in the metal may nowhere exceed 4,000-lbs.

(34).–The teeth of a cast-iron wheel are $3\frac{1}{2}$-ins. long, $2\frac{1}{4}$-ins. deep, and 7-ins. wide; what is the breaking weight of a tooth? (C=5,000-lbs.)

(35).–A wrought iron bar 4-ins. deep, $\frac{3}{4}$-in. wide, and rigidly fixed at one end, gave way, when loaded with 1,568-lbs. at the free end, at a point 2-ft. 8-ins. from the load; find C.

(36).–Determine the diameter of a solid wrought iron round beam which rests upon supports 5-ft. apart, and is about to give way under a load of 30-tons at 14-ins. from one end. (C=5000-lbs.).

(37.)–A timber beam 6-ins. deep, 3-ins. wide, and weighing 50-lbs. per cubic foot, was placed upon supports 8-ft. apart, and broke down under a weight of 10,000-lbs. at the centre; find C.

(38).–A wrought iron bar, 2-ins. wide and 4-ins. deep, rests upon supports 12-ft. apart; determine the uniformly distributed load which the bar will safely carry in addition to its own weight. (C=50,000-lbs., and factor of safety = 4).

(39).–Find the length of a beam of Canadian Ash 6-ins. square, which would break of its own weight, when supported at the ends (the weight of the timber = 30-lbs. per cubic ft.).

(40).–A wrought iron bar 20-ft. in length and $1\frac{1}{2}$-ins. in width is fixed at one end; find the depth of the bar so that it might safely carry its own weight together with 500-lbs. at the free end.

(41).–A railway girder, 50-ft. in the clear, and 6-ft. deep, carries a uniformly distributed load of 50-tons; find the maximum shearing stress at 20-ft. from one end, when a train weighing $1\frac{1}{4}$-tons per lineal ft. crosses the girder.

Also, find the minimum theoretic thickness of the web., 4-tons being the safe shearing inch stress of the metal.

(42).–The flanges of a rolled joist are each 4-ins. wide and $\frac{1}{2}$-in. thick; the web is 8-ins. deep between the flanges and $\frac{1}{2}$-in. thick; find the position of the neutral axis, the inch-stresses in the metal being 10,000–lbs. in tension and 8,000–lbs. in compression.

(43).–A wrought iron semi-girder is 7-ft. long 12-ins. deep, and its flanges are each 4-ins. wide and $\frac{1}{2}$-in. thick; find the weight at the centre which will cause the upper flange to fail, the ultimate tensile inch-stress of wrought iron being 20–tons.

(44).–A continuous lattice girder is supported at four points, each of the side spans being 140-ft. 11-ins. in length, 22-ft. 3 ins. in depth, and weighing .68-tons per lineal ft. On one occasion an excessive load on the centre span lifted the end of one of the side spans off the abutunent; find the consequent inch-stress in the bottom flange at the pier, where its gross sectional area is 127-sq. ins.

(45).–A railway girder is 101·2-ft. in the clear, 22·25-ft. deep, and weighs 3764–lbs. per lineal ft.; find the maximum shearing and flange stresses at 25-ft. from one end, when a train weighing 2500–lbs. per lineal ft. crosses the girder.

(46).–A bridge on the South Staffordshire railway is supported upon two main girders, each 51-ft. 4-ins. in the clear, 6-ft. 6-ins. deep at the centre, and 4-ft. deep at each end. The uniformly distributed dead load upon the girders is 43-tons; the gross sectional areas of the top and bottom flanges at the centre are 27-sq. ins. and 28-sq-ins. respectively; find the corresponding inch-stresses, assuming the efficiency of the tension flange to be reduced *one-fifth* by riveting.

Determine the uniformly distributed rolling load which will increase the inch-stresses by 2-tons.

(47).–A lattice girder of 80-ft. span and 8-ft. deep is designed to carry a dead load of 60-tons and a live load of 120-tons. uniformly distributed. At the centre the *net* sectional area of the bottom flange is 45-sq.-ins., and the *gross* sectional area of the top flange is $56\frac{1}{2}$-sq. ins.; find the corresponding inch-stresses and the position of the neutral axis.

(48).–A cast-iron semi-girder, 8-ft. long and 12-ins. deep, carries a uniformly distributed load of 16,000–lbs.; find the area of the top flange at the support, so that the stress in the metal may not exceed 3000-lbs. per sq. in.

(49).–A cast-iron girder, $27\frac{1}{2}$-ins. deep, rests upon supports 26-ft. apart. Its bottom flange is 16-ins. wide and 3-ins. thick; find the breaking weight at the centre, the tearing inch-stress of cast-iron being 15,000-lbs. (Neglect the web).

(50).–A plate girder, 64-ft. in the clear and 8-ft. deep, carries a dead load of 2-tons per lineal ft. At any vertical section the two flanges are of equal area, and their joint area is equal to that of the web; find the sectional area at the centre of the girder, so that the stress in the metal may not exceed 3-tons per sq. in. The deflection of the girder is $\frac{3}{8}$-in. at the centre, find the corresponding curvature, and also E.

(51).–The crushing and tearing inch-stresses of cast-iron are 80,000-lbs. and 16,000-lbs., respectively; shew that, approximately, in a well-proportioned T-section, $A_1 = 2 \cdot A_2$, and in a well-proportioned double T-section, $A_3 = 5 \cdot A_1 + 2 \cdot A_2$, (§(8)).

Find the moment of resistance, the depth of the section being h.

(52).–The crushing and tearing inch-stresses of wrought-iron are 40,000-lbs. and 60,000-lbs. respectively; shew that, approximately, in a well-proportioned T-section, $A_1 = \frac{A_2}{6}$, and in a well proportioned double T-section, $A_3 = \frac{2}{3} \cdot A_1 - \frac{A_2}{6}$.

Find the moment of resistance, the depth of the section being h.

(53).–If the ultimate compressive strength of a metal is equal to its ultimate tensile strength, shew that a rectangle should be substituted for a T-section.

(54).–The effective length and depth of a cast iron girder were $27\frac{1}{3}$ ft. and 18-ins. respectively, and its bottom flange was 10-ins. wide and $1\frac{1}{2}$-ins. thick. The girder failed under a weight of $29\frac{1}{2}$-tons at the centre, what was the maximum inch. stress in the bottom flange? (Neglect the aid from a $\frac{3}{4}$-in. web).

(55).–The flanges of a girder are of equal sectional area and their joint area is equal to that of the web; what must be the sectional area of the girder to resist a bending moment of 300-inch tons, the effective depth being 10-ins., and the limiting inch-stress in the metal 4-tons?

(56).–The length of the Conway tubular bridge from centre to centre of bearings is 412-ft.; the effective depths of a tube at the centre and quarter spans are 23·7-ft. and 22.25-ft., respectively; the corresponding sectional areas of the top and bottom flanges are 645-sq-ins., 536-sq-ins., and 566-sq.-ins., 461-sq-ins.; the corresponding sectional areas of the web are 257-sq.-ins. and 241-sq-ins.; assuming that the total dead load upon a tube is equivalent to 3-tons per lineal ft., and that the continuity of the web compensates for the weakening of the tension flange by riveting, find the flange stresses and deflections at the centre and quarter spans, E being 24,000,000-lbs. What will be the increase in the central flange stresses and deflections under a uniformly distributed live load of $\frac{3}{4}$-ton per lineal ft.?

(57).–Find the central flange stresses and the E of the metal of one of the Britannia tubes from the following data :–

Effective length $=470$-ft., effective depth$=27\frac{1}{2}$-ft., uniformly distributed dead load upon tube$=1587$-tons, the deflection at the centre $=$ 12-ins., the gross sectional area of the top and bottom flanges and of the web, at the centre$=648$-sq.-ins., 585-sq.-ins., 302-sq.-ins., respectively.

(58).–The effective length and depth of a cast-iron girder were 57-ins. and $5\frac{1}{8}$-in., respectively; the top flange was 2·33-ins. wide and ·31-ins. thick; the bottom flange was 6·67-ins. wide and ·66-ins. thick; the web was ·266-ins. thick. The girder failed under a load of 18-tons at the centre, what were the maximum inch-stresses in each flange at the moment of rupture?

(59).–The effective length and depth of a cast-iron girder were 11-ft. 7-ins. and 10-ins. respectively; the top flange was $2\frac{1}{2}$-ins. wide and $\frac{7}{8}$-in. thick; the bottom flange was 10-ins. wide and $1\frac{1}{4}$-ins. thick; the web was $\frac{3}{4}$-in. thick. The girder was tested by loading it at two points distant $3\frac{3}{4}$-ft. from each end, and failed under a load of $17\frac{1}{2}$-tons at each point; what were the central flange stresses at the moment of rupture? Find the deflection at the centre, when the load at each of the points was $7\frac{1}{2}$-tons. (E=18,000,000-lbs., and weight of girder=3,368-lbs).

(60).–A cylindrical beam 2-ins. in diameter, 60-ins. in length and weighing $\frac{1}{4}$-lb. per cubic inch, deflects $\frac{3}{8}$-in. under a weight of 3,000-lbs. at the centre. Find E.

(61).–Compare the *stiffness* of the beams in question (22), and also of the beams in question (27). If these beams are all of the same material, and if the rectangular and square beams have the same sectional area, in what relative order will they fail under a blow at the centre?

(62).–A stress of 1-lb. per sq.-in. produces a strain of $\frac{1}{2,000,000}$ in a beam 12-ins. square, resting upon supports 20-ft. apart; find the radius of curvature and maximum deflection under a central load of 2,000-lbs.

(63).–A piece of greenheart 11-ft. 9-ins. long, 11-ft. 7-ins. between bearings, 9-ins. wide and 8-ins. deep, was successively subjected to loads of 4, 8, and 16-tons, at the centre, the corresponding deflections being, ·32-ins., ·64-ins., and 1·28-ins.; find E and the total work done in bending the beam. What were the corresponding inch-stresses at $\frac{3}{4}$ths the depth of the beam?

(64).–Shew that the *sudden* application of a load to a structure produces a greater deflection than the *gradual* application of the same load; also that the deflection of a beam in which the elastic limit is not exceeded is twice as great from the suddenly applied load as from the gradually applied load.

(65).–Shew that the capability of a rectangular beam to resist a blow in a direction transverse to its length is independent of the proportion of the depth to the breadth.

(66).–The length, depth, and weight of one of the Victoria Bridge tubes are 242-ft., 19-ft., and 275-tons, respectively; the inch-stress in the metal from its own weight does not exceed 2-tons; find the length, depth, and weight of a similar tube which has to bear an inch-stress of 4-tons. The deflection of the first tube from its own weight was 2-ins., what will be the deflection of the similar tube?

(67).–If the live load in question (47) cross the girder at 60 miles an hour, what will be the increased pressure due to centrifugal force?

(68).–Sketch the cast-iron girders which may be employed for a 50-ft. railway bridge, giving the section at the centre and near the ends.

(69).–Determine the relations between p, q, r and n, [§ (16)], so that two beams of the same material may be subjected to equal unit-stresses at corresponding points.

Shew that the ratio of the deflections of the beams is p.

(70).–Two equally loaded rectangular girders are of the same length, while the breadth and depth of the one are respectively equal to the depth and breadth of the other; shew that the ratio of unit-stresses due to bending is q, of the unit-stresses due to shearing is *unity*, of the deflections is q^2.

(71).–A straight wrought-iron bar is capable of sustaining as a strut a weight w_1, and a weight w_2 as a beam supported at the ends and loaded in the middle, the beam being of such proportions that its deflection is small compared with its thickness; if the bar has to sustain a weight w as a strut and a weight w' as a beam, simultaneously, shew that it will not break if $w + \frac{w_1}{w_2}.w' < w_1$.

(72).–A floor, with superimposed load, weighs 140-lbs. per sq.-ft., and is carried by a girder 50-ft. between bearings. What additional weight per sq-ft. should be added for the weight of the girder?

(73).–A floor, with superimposed load, weighs 160-lbs. per sq.-ft., and is carried by tubular girders 17-ft. centre to centre and 42-ft. between bearings. Determine the depth of the girders, the safe tensile inch-stress in the metal being 9000-lbs.

(74).–A cast-iron girder is 20-ft. long between bearings and 24-ins. deep at the centre, where a weight of 30,000-lbs. is concentrated; determine all the dimensions at the central section, 5 being the factor of safety.

(75).–The length of a cast-iron girder between bearings is 25-ft., and the sectional area of the bottom flange is 36-sq. ins.; find the depth of the girder so that it may safely carry a load of 80,000-lbs. concentrated at 10-ft. from one end, 5 being the factor of safety.

(76).–An arched girder is 25-ft. between bearings, is 36-ins. deep at the centre, and carries a uniformly distributed load of 100,000-lbs.; find the sectional area of the horizontal tie, which is to be of wrought-iron.

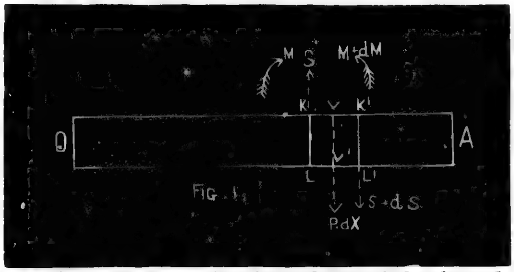
(77).–Determine the admissible flange unit-stresses, in Questions 45, 47 and 53, according to the principles of § (13) Chap. I.

Read this Chap. for Honours

CHAPTER B.

THE EQUILIBRIUM AND STRENGTH OF BEAMS.

(1).–*General Equations.*–The girder OA of length l carries a load of which the intensity varies *continuously* and is p at a point K distant x from O.



Consider the conditions of equilibrium of a slice of the girder bounded by the vertical planes KL, $K'L'$, of which the abscissæ are $x, x+dx$, respectively.

The load between these planes may, without sensible error, be supposed to be uniformly distributed, and its resultant $p.dx$ therefore acts along the centre line VV'.

The forces acting upon the slice at the plane KL are equivalent to an *upward* shearing force S, and a *right-handed* couple of which the moment is M, while the forces acting upon the slice at the plane $K'L'$ are equivalent to a *downward* shearing force $S+dS$, and a *left-handed* couple of which the moment is $M+dM$.

Since there is to be equilibrium,

$S-(S+dS)-p.dx=$ the algebraic sum of the vertical forces $=0$.

$$\therefore \frac{dS}{dx}+p=0. \qquad (a).$$

And, $M-(M+dM)+S.\dfrac{dx}{2}+(S+dS).\dfrac{dx}{2}=$ the algebraic sum of the moments of the forces with respect to V or $V'=0$.

$$\therefore \frac{dM}{dx}-S=0. \qquad (b).$$

The term $\dfrac{dS.dx}{2}$ is disregarded, being indefinitely small as compared with the remaining terms.

Equations (a) and (b) are the general equations applicable to girders carrying loads of which the intensity varies *continuously*. Their integration is easy, and introduces two arbitrary constants which are to be determined in each particular case.

Cor. 1.–From equations (a) and (b), $\frac{d^2M}{dx^2}=\frac{dS}{dx}=-p$

let $p=w.f(x)$, w being a constant, and $f(x)$ some function of x.

$$\therefore \frac{dM}{dx}=c_1-w.\int_0^x f.(x).dx. \text{ and } M=c_2+c_1.x-w.\int_0^x\int_0^x f(x).dx^2$$

c_1 and c_2 being the constants of integration, and 0 and x the limits.

Ex.–Let the girder rest upon two supports, and carry a uniformly distributed load of intensity w_1,

$$\therefore \frac{dM}{dx}=c_1-\int_0^x w_1.dx=c_1-w_1.x, \text{ and } M=c_2+c_1.x-w_1.\frac{x^2}{2}$$

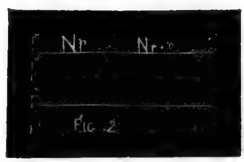
But M is zero, when $x=0$ and also when $x=l$, $\therefore c_2=0$ and $c_1=\frac{w_1.l}{2}$

$$\text{Hence, } M=\frac{w_1l}{2}.x-\frac{w_1}{2}.x^2$$

$$\text{and } S=\frac{dM}{dx}=\frac{w_1.l}{2}-w_1.x.$$

Cor. 2.–The bending moment is a maximum at the point defined by $\frac{dM}{dx}=0=S$, i.e., at a point at which the shearing force vanishes.

In the preceding example, the position of the maximum bending moment is given by $S=0=\frac{w_1.l}{2}-w_1.x$, or $x=\frac{l}{2}$, and its corresponding value is $\frac{w_1.l}{2}.\frac{l}{2}-\frac{w_1}{2}.\frac{l^2}{4}=\frac{w_1.l^2}{8}$.



Cor. 3.–Suppose that the load, instead of varying continuously, consists of a number of finite weights at isolated points.

By reason of the discontinuity of the loading, the general equations can only be integrated between consecutive points.

Let N_r, N_{r+1}, be any two such points, of abscissæ x_r, x_{r+1}, respectively.

Between these points Equations (a) and (b) become,

$$\frac{dS}{dx}=0, \text{ and } \frac{dM}{dx}=S.$$

$\therefore S=$ a constant $=S_r$, suppose, between N_r and N_{r+1}.

Hence, $\dfrac{dM}{dx}=S_r$, and $M=S_r.x+C$, between N_r and N_{r+1}, C being a constant of integration.

Let $M=M_r$, when $x=x_r$, $\therefore C=M_r-S_r.x_r$, and $M=S_r.(x-x_r)+M_r$.

Also, if $M=M_{r+1}$, when $x=x_{r+1}$, $\therefore M_{r+1}=S_r.(x_{r+1}-x_r)+M_r$

The terminal conditions will give additional equations, by means of which the solution may be completed.

Ex. – The girder OA rests upon two supports at O, A, and carries weights P_1, P_2, at points B, C, dividing the girder into three segments, OB, BC, CA, of which the lengths are r, s, t, respectively.

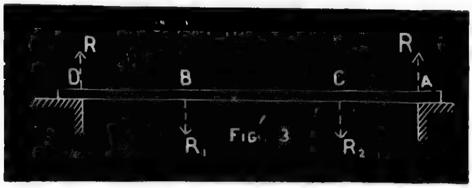


Fig. 3

The reaction R_1 at $O=\dfrac{P_1.\overline{s+t}+P_2.t}{l}$,

The reaction R_2 at $A=\dfrac{P_1.r+P_2.\overline{r+s}}{l}$.

Between O and B, S is constant$=S_r$, suppose, $=R_1$;

$\therefore M=S_r.x$, there being no constant of integration as $M=0$ when $x=0$.

Also, when $x=r$, $M=S_r.r$

Between B and C, S is constant$=S_s$, suppose, $=R_1-P_1$

$\therefore M=S_s.x+c'$, c' being the constant of integration.

But $M=S_r.r$, when $x=r$, $\therefore c'=(S_r-S_s).r$

and $M=S_s.x+(S_r-S_s).r$

Also, when $x=r+s$, $M=S_s.s+S_r.r$

Between C and A, S is constant$=S_t$, suppose$=R_1-P_1-P_2$.

and $\therefore M=S_t.x+c''$, c'' being the constant of integration.

But $M=S_s.s+S_r.r$, when $x=r+s$, $\therefore c''=S_s.s+S_r.r-S_t.(r+s)$;

and $M=S_t.x+S_s.s+S_r.r-S_t.(r+s)$.

Hence, at A, $0=S_t.t+S_s.s+S_r.r$.

Cor. 4. – The Equation $\dfrac{dM}{dx}=S$, indicates that the shearing force at vertical section of a girder is the *increment* of the bending moment at that section per unit of length, and is an important relation in calculating the number of rivets required for flange and web connections.

(2).—*On the distribution of stress in any given plane of a loaded rectangular beam under combined bending and shearing actions.*— $abcd$ is an indefinitely small rectangular element of the beam bounded by the planes ab, cd, parallel to the neutral axis, NN, and by the vertical planes, ad, bc.

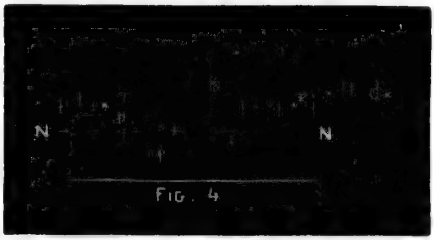


FIG. 4

Let $ab=dx$, $ad=dy$, and let the thickness of the beam perpendicular to the plane of the paper be unity.

Let s be the intensity of the downward shearing stress along ad.
$\therefore s+ds$ is " " " upward " " cb.
Let t be the intensity of the tangential (shearing) stress along cd.
$\therefore t+dt$ is " " " " " " ab.
Let p be the intensity of the compressive stress perpendicular to ad.
$\therefore p+dp$ is " " " " " " bc.

The intensities are assumed to be the same at every point of the faces to which they respectively refer, as the element $a\,b\,c\,d$ is extremely small.

The total compression upon $bc=(p+dp).dy=p.dy$, nearly, and therefore balances the total compression upon ad.

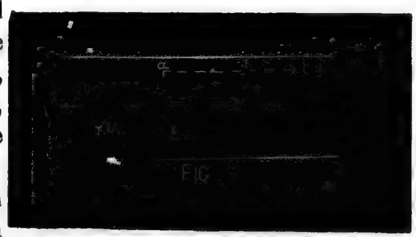
The total shear upon $bc=(s+ds).dy=s.dy$, nearly, and therefore forms with the shear upon ad a couple of moment $s.dy.dx$.

The total shear upon $ab=(t+dt).\ dx=t.dx$, nearly, and therefore forms with the shear upon cd, a couple of moment $t.dx.dy$.

Since there is equilibrium, these couples must be equal and opposite and $\therefore t.dx.dy=s.dy.dx$, or $t=s$.

Hence, the tangential intensity, i.e., the intensity of the resistance to a longitudinal shear, at any point, is equal in magnitude to the intensity of the vertical shear at the same point.

Consider an indefinitely small triangular element abc of the beam, bounded by a plane bc, inclined at θ to the neutral axis, the horizontal plane ab, and the vertical plane ac.



The element abc is kept in equilibrium by the compression

$p.ac$ upon ac, the shear $t.ab(=s.ab)$ along ab, the shear $s.ac$ along ac, and the stress developed in the plane bc.

Let the stress upon bc be decomposed into two components, the one $X.bc$ normal to bc, the other $Y.bc$ tangential to bc.

Resolve in directions perpendicular and parallel to bc;

$$\therefore X.bc = p.ac.sin\theta + s.ab.sin\theta + s.ac.cos\theta$$

$$\text{and } Y.bc = p.ac.cos\theta + s.ab.sin\theta - s.ac.sin\theta$$

$$\text{or, } X = p.sin^2\theta + s.sin2\theta.$$

$$\text{and, } Y = \frac{p.sin2\theta}{2} + s.cos2\theta$$

Let θ_1, θ_2, be the values of θ, for which X and Y, respectively, are maxima.

$$\therefore \frac{dX}{d\theta} = 0 = p.sin2\theta_1 + 2s.cos.2\theta_1, \text{ and } tan.2\theta_1 = -\frac{2s}{p}.$$

$$\frac{dY}{d\theta} = 0 = p.\,cos2\theta_2 - 2.s.sin2\theta_2, \text{ and } tan2\theta_2 = \frac{p}{2s}.$$

$$\therefore tan2\theta_1.tan\ 2\theta_2 = -1, \text{ and } \theta_1 - \theta_2 = 45^\circ.$$

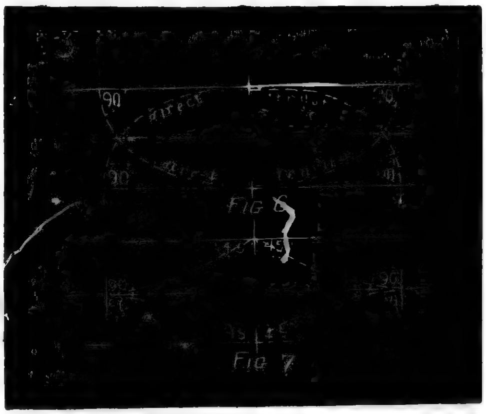
Hence, at any point, the angle between the plane upon which the normal intensity of stress is a maximum, and the plane upon which the tangential intensity of stress is a maximum, is equal to 45°.

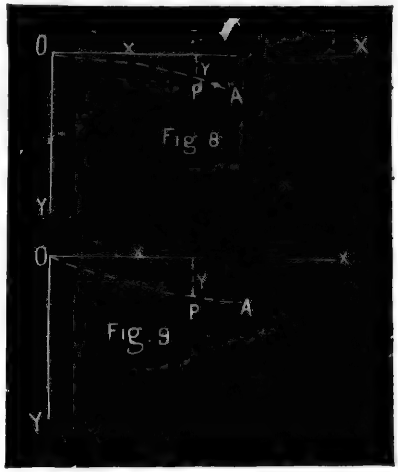
Again, s is zero when $\theta_1 = 90^\circ$ or 0°, and p is zero when $\theta_1 = 45^\circ$.

Thus, the *curve of greatest normal intensity* cuts the neutral axis at an angle of 45°, one outside fibre at 90°, and the opposite outside fibre at 0°, while the *curve of greatest tangential intensity* cuts the outside fibre at 45°, and touches the neutral axis.

Fig. 6 serves to illustrate the curves of greatest normal intensity There are evidently two sets of these curves, referring, respectively, to direct *thrust* and direc *tension*.

Fig. 7 illustrates the curves of greatest tangential intensity.





(3).—*On the interpretation of the Bending Moment equation.*—The bending moment M at any transverse section of a girder may be obtained from the equation, $M = \frac{E.I}{R}$, R being the radius of curvature of the neutral axis at the section under consideration.

Let OA, in Figs. 8 and 9, represent a portion of the neutral axis of a bent girder.

Ta.. O as the origin, the horizont l line OX as the axis of x, and the line OY drawn vertically downwards as the axis of y.

Let x, y, be the co-ordinates of any point P in the neutral axis.

If R is the radius of curvature at P, $\therefore \frac{1}{R} = \pm . \frac{\frac{d^2.y}{dx^2}}{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}$

the sign being + or − according as the girder is bent, as in Fig. 8 or as in Fig. 9.

Now, $\frac{dy}{dx}$ is the tangent of the angle which the tangent line at P to the neutral axis makes with OX, and the angle is always very small. Thus, $\frac{dy}{dx}$ is also very small, and squares and higher powers of $\frac{dy}{dx}$ may be disregarded without serious error.

$\therefore \frac{1}{R} = \pm \frac{d^2y}{dx^2}$, and the bending moment equation becomes,

$$M = \pm . E.I. \frac{d^2y}{dx^2}.$$

The integration of this equation introduces two arbitrary constants, of which the values are to be determined from given conditions.

(4).—*Examples of the form assumed by the neutral axis of a loaded beam.*—

Ex. 1.–A semi-girder fixed at one end O carries a weight W at the other end A. At any point P, (x,y), of the neutral axis,

$$+E.I.\frac{d^2y}{dx^2}=W.(l-x). \qquad \textbf{(A)}$$

Integrating, $E.I.\frac{dy}{dx}=W.\left(l.x-\frac{x^2}{2}\right)+c_1$, c_1 being a constant of integration. But the girder is fixed at O, so that the inclination of the neutral axis to the horizon at this point is zero, and thus, when $x=0$, $\frac{dy}{dx}$ is 0, and $\therefore c_1=0$.

$$\text{Hence, } E.I.\frac{dy}{dx}=W.\left(l.x-\frac{x^2}{2}\right) \qquad \textbf{(B)}$$

Integrating, $E.I.y=W.\left(l.\frac{x^2}{2}-\frac{x^3}{6}\right)+c_2$, c_2 being a constant of integration. But $y=0$, when $x=0$, and $\therefore c_2=0$.

$$\text{Hence, } E.I.y=W\left(l.\frac{x^2}{2}-\frac{x^3}{6}\right) \qquad \textbf{(C)}$$

Equation (B) gives the value of $\frac{dy}{dx}$, *i.e.*, the *slope*, at any point of which the abscissa is x.

Equation (C) defines the curve assumed by the neutral axis, and gives the value of y, *i.e.*, the *deflection*, corresponding to any abscissa x.

Cor.–Let a_1 be the slope, and d_1 the deflection at A.

$$\therefore \text{from } B,\ tan\ a_1=\frac{1}{2}\cdot\frac{W.l^2}{E.I}, \text{ and from } (C),\ d_1=\frac{1}{3}\cdot\frac{W.l^3}{E.I}$$

Ex. 2.–A semi-girder fixed at one end O carries a uniformly distributed load of intensity w.

At any point P, (x,y), of the neutral axis,

$$+E.I.\frac{d^2y}{dx^2}=\frac{w}{2}.(l-x)^2=\frac{w}{2}\cdot(l^2-2.l.x+x^2) \qquad \textbf{(A)}$$

Integrating, $+E.I.\frac{dy}{dx}=\frac{w}{2}\cdot\left(l^2.x-l.x^2+\frac{x^3}{3}\right)+c_1$, c_1 being a constant of integration.

$$\text{But } \frac{dy}{dx}=0, \text{ when } x=0, \text{ and } \therefore\ c_1=0$$

$$\text{Hence, } E.I.\frac{dy}{dx}=\frac{w}{2}.\left(l^2.x-l.x^2+\frac{x^3}{3}\right). \tag{B}$$

Integrating, $E.I.y=\frac{w}{2}.\left(l^2.\frac{x^2}{2}-l.\frac{x^3}{3}+\frac{x^4}{12}\right)+c_2$, c_2 being a constant of integration.

$$\text{But } y=0, \text{ when } x=0, \text{ and } \therefore\ c_2=0.$$

$$\text{Hence, } E.I.y=\frac{w}{2}.\left(l^2.\frac{x^2}{2}-l.\frac{x^3}{3}+\frac{x^4}{12}\right). \tag{C}$$

Cor.–Let a_2 be the slope and d_2 the deflection at A,

$$\therefore \text{ from } B,\ tan\ a_2=\frac{1}{6}.\frac{w.l^3}{E.I}., \text{ and from } C,\ d_2=\frac{1}{8}.\frac{w.l^4}{E.I}.$$

Ex. 3.–A semi-girder fixed at one end carries a uniformly distributed load of intensity w, and also a single weight W at the free end. This is merely a combination of Examples (1) and (2), and the resulting equations are:–

$$E.I.\frac{d^2y}{dx^2}=W.(l-x)+\frac{w}{2}.(l-x)^2 \tag{A}$$

$$E.I.\frac{dy}{dx}=W.\left(l.x-\frac{x^2}{2}\right)+\frac{w}{2}.\left(l^2.x-l.x^2+\frac{x^3}{3}\right). \tag{B}$$

$$E.I.y=W.\left(l.\frac{x^2}{2}-\frac{x^3}{6}\right)+\frac{w}{2}.\left(l^2.\frac{x^2}{2}-l.\frac{x^3}{3}+\frac{x^4}{12}\right). \tag{C}$$

Also, if A is the slope, and D the deflection, at the free end,

$$\therefore \text{ from } B,\ tan\ A=\frac{1}{E.I}.\left(\frac{W.l^2}{2}+\frac{w.l^3}{6}\right)=tan\ a_1+tan\ a_2.$$

$$\text{and from } C,\ D=\frac{1}{E.I}\left(\frac{W.l^3}{3}+\frac{w.l^4}{8}\right)=d_1+d_2.$$

Ex. 4.–The girder OA rests upon two supports at O, A, and carries a weight W at the centre.

The neutral axis is evidently symmetrical with respect to the middle point C, and at any point P, (x,y), *between* O and C,

$$-E.I.\frac{d^2y}{dx^2}=\frac{W}{2}.x \tag{A}$$

Integrating, $-E.I.\frac{dy}{dx}=\frac{W}{4}.x^2+c_1$, c_1 being a constant of integration.

But the tangent to the neutral axis at C must be horizontal, so that when $x=\frac{l}{2}$, $\frac{dy}{dx}=0$, and $\therefore c_1=-\frac{W.l^2}{16}$

$$\text{Hence,}\quad -E.I.\frac{dy}{dx}=\frac{W}{4}.x^2-\frac{W.l^2}{16} \qquad \text{(B)}$$

Integrating, $-E.I.y=\frac{W}{12}.x^3-\frac{W.l^2}{16}.x+c_2$, c_2 being a constant of integration.

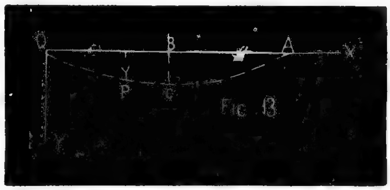
But $y=0$, when $x=0$, and $\therefore c_2=0$

$$\text{Hence,}\quad -E.I.y=\frac{W}{12}.x^3-\frac{W.l^2}{16}.x \qquad \text{(C)}$$

Cor.—Let a_1 be the slope at O, and d_1 the deflection at the centre,

$\therefore$ from (B), $\tan a_1=\frac{1}{16}.\frac{W.l^2}{E.I}$, and from (C), $d_1=\frac{1}{48}.\frac{W.l^3}{E.I}$.

Ex. 5.—The girder OA rests upon supports at O, A, and carries a uniformly distributed load of intensity w.



At any point P, (x,y), of the neutral axis,

$$-E.I.\frac{d^2y}{dx^2}=\frac{w.l}{2}.x-\frac{w.x^2}{2} \qquad \text{(A)}$$

Integrating, $-E.I.\frac{dy}{dx}=\frac{w.l}{4}.x^2-\frac{w.x^3}{6}+c_1$, c_1 being a constant of integration.

But, $\frac{dy}{dx}=0$, when $x=\frac{l}{2}$, and $\therefore c_1=-\frac{w.l^3}{24}$

$$\text{Hence,}\quad -E.I.\frac{dy}{dx}=\frac{w.l}{4}.x^2-\frac{w.x^3}{6}-\frac{w.l^3}{24} \qquad \text{(B)}$$

Integrating, $-E.I.y=\frac{w.l}{12}.x^3-\frac{w.x^4}{24}-\frac{w.l^3}{24}.x+c_2$, c_2 being a constant of integration.

But $y=0$, when $x=0$, and $\therefore c_2=0$

$$\text{Hence,}\quad -E.I.y=\frac{w.l}{12}.x^3-\frac{w.x^4}{24}-\frac{.l^3}{24}.x \qquad \text{(C)}$$

Cor.—Let a_2 be the slope at O, and d_2 the deflection at the centre,

$\therefore$ from (*B*), $\tan a_2=\frac{1}{24}.\frac{w.l^3}{E.I}$, and from (*C*), $d_2=\frac{5}{384}.\frac{w.l^4}{E.I}$

Ex. 6.–A girder rests upon two supports, and carries a uniformly distributed load of intensity w, together with a single weight W at the centre. This is merely a combination of Examples (4) and (5), and the resulting equations are :–

$$-E.I.\frac{d^2y}{dx^2}=\frac{W}{2}.x+\frac{w.l}{2}.x-\frac{w.x^2}{2} \qquad \text{(A)}$$

$$-E.I.\frac{dy}{dx}=\frac{W}{4}.x^2-\frac{W}{16}.l^2+\frac{w.l}{4}.x^2-\frac{w.x^3}{6}-\frac{w.l^3}{24} \qquad \text{(B)}$$

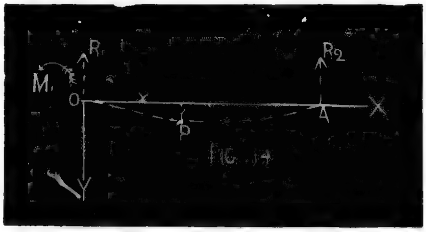
$$\text{and } -E.I.y=\frac{W}{12}.x^3-\frac{W}{16}.l^2.x+\frac{w.l}{12}.x^3-\frac{w.x^4}{24}-\frac{w.l^3}{24}.x \qquad \text{(C)}$$

Also, if A is the slope at the origin, and D the central deflection,

$$\therefore \text{ from } (B),\ tan\ A=\frac{1}{E.I}.\left(\frac{W.l^2}{16}+\frac{w.l^3}{24}\right)=tan\ a_1,+tan\ a_2.$$

$$\text{and from } (C),\ D=\frac{1}{E.I}.\left(\frac{W.l^3}{48}+\frac{5}{384}.w.l^4\right)=d_1+d_2.$$

Ex. 7.–Suppose that the end O of the girder in Example (5) is *fixed*. The fixture introduces a *left-handed* couple at O; let its moment be M_1



Let the reactions at O and A, be R_1, R_2, respectively.

At any point P, (x,y) of the neutral axis,

$$-E.I.\frac{d^2y}{dx^2}=R_1.x-\frac{w\,x^2}{2}-M_1 \qquad (1)$$

But M, *i.e.*, $-E.I.\frac{d^2y}{dx^2}$, is zero when $x=l$,

$$\therefore M=R.l-\frac{w.l^2}{2} \qquad (2)$$

$$\text{and, } -E.I.\frac{d^2y}{dx^2}=R.(x-l)-\frac{w}{2}.(x^2-l^2) \qquad (3)$$

$$\text{Integrating, } -E.I.\frac{dy}{dx}=R.\left(\frac{x^2}{2}-l.x\right)-\frac{w}{2}.\left(\frac{x^3}{3}-l^2.x\right) \qquad (4)$$

There is no constant of integration as $\frac{dy}{dx}$ and x vanish together.

$$\text{Integrating again, } -E.I.y=R.\left(\frac{x^3}{6}-\frac{l.x^2}{2}\right)-\frac{w}{2}.\left(\frac{x^4}{12}-\frac{l^2.x^2}{2}\right) \qquad (5)$$

There is no constant of integration as x and y vanish together.

But y also vanishes when $x=l$, so that $0=-R_1.\frac{l^3}{3}+\frac{5}{24}.w.l^4$

$$\therefore\ R_1=\frac{5}{8}.w.l, \text{ and so } R_2=\frac{3}{8}.w.l \qquad (6)$$

$$\text{From equation (2), } M_1=\frac{w.l^2}{8} \qquad (7)$$

Hence, the *bending moment*, *slope*, and *defln* equations are respectively,

$$-E.I.\frac{d^2y}{dx^2}=\frac{5}{8}.w.l.x-\frac{w}{2}.x^2-\frac{w.l^2}{8}=M \qquad (8)$$

$$-E.I.\frac{dy}{dx}=\frac{5}{16}.w.l.x^2-\frac{w}{6}.x^3-\frac{w.l^2}{8}.x \qquad (9)$$

$$-E.I.y=\frac{5}{48}.w.l.x^3-\frac{w}{24}.x^4-\frac{w.l^2}{16}.x^2 \qquad (10)$$

Cor. 1.–The bending moment vanishes when

$$M=0=\frac{5}{8}.w.l.x-\frac{w}{2}.x^2-\frac{w.l^2}{8},$$

i.e., when $x=\frac{l}{4}$ or l. Between these limits the Bending Moment is positive, and is a maximum when $\frac{dM}{dx}=0=\frac{5}{8}.w.l-w.x$, i.e., when $x=\frac{5}{8}.l$, its value being $\frac{9}{128}.w.l^2$.

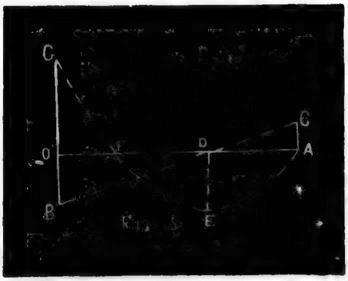
Between the limits $x=0$ and $x=\frac{l}{4}$, the Bending Moment is negative, and at O is an absolute maximum, its value being $-\frac{w.l^2}{8}$.

Cor. 2.–The deflection is a maximum when $\frac{dy}{dx}=0$, i.e., when

$$0=\frac{5}{16}.w.l.x^2-\frac{w}{6}.x^3-\frac{w.l^2}{8}.x, \text{ or } x=\frac{l}{16}\cdot\left(15-\sqrt{33}\right)$$

The corresponding value of y is found by substituting this value of x in equation (10).

Cor. 3.–To illustrate the Shearing Force and Bending Moment at different points of the girder, *graphically*.



The shearing force at any point of which the abscissa is x, is

$$S=\frac{5}{8}.w.l-w.x \qquad (11)$$

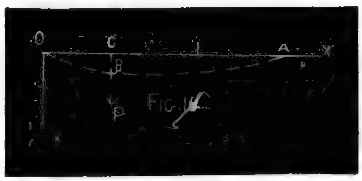
The Bending Moment at the same point is,

$$M=\frac{5}{8}.w.l.x-\frac{w}{2}.x^2-\frac{w.l^2}{8} \qquad (12)$$

Take OB and AC, respectively equal or proportional to $\frac{5}{8}.w.l.$ and $\frac{3}{8}.w.l$; join BC. The line BC cuts OA in D, where $OD=\frac{5}{8}.l$. The shearing force at any point is represented by the ordinate between that point and the line BC.

Again, take OG, DE, and OF, respectively equal or proportional to $\frac{w.l^2}{8}$, $\frac{9}{128}.w.l^2$, and $\frac{l}{4}$. The bending moment at any point is represented by the ordinate between that point and the parabola passing through G, F, and A, having its vertex at E and its axis vertical.

(5).—*To discuss the form assumed by the neutral axis of a girder OA which rests upon supports at O and A, and carries a weight P at a point B, distant r from O.*



Let OBA be the neutral axis of the deflected girder.

The reactions at O and A, are $P.\frac{l-r}{l}$, and $P.\frac{r}{l}$, respectively.

Let BC, the deflection at C, $=d$.

Let a be the slope of the neutral axis at B.

The portions OB, BA, must be treated separately as the weight at B causes discontinuity in the equation of moments.

First, at any point (x,y) of OB,

$$-E.I.\frac{d^2y}{dx^2}=P.\frac{l-r}{l}.x \qquad (1)$$

Integrating, $-E.I.\frac{dy}{dx}=P.\frac{l-r}{l}.\frac{x^2}{2}+c_1$, c_1 being a constant of integration.

But $\frac{dy}{dx}=tan\ a$, when $x=r$, and $\therefore -E.I.\ tan\ a=P.\frac{l-r}{l}.\frac{r^2}{2}+c_1$

$$\text{Hence, } -E.I.\left(\frac{dy}{dx}-tan\ a\right)=P.\frac{l-r}{l}.\left(\frac{x^2}{2}-\frac{r^2}{2}\right) \qquad (2)$$

$$\text{Integrating, } -E.I.(y-x\ tan\ a)=P.\frac{l-r}{l}.\left(\frac{x^3}{6}-\frac{r^2}{2}.x\right) \qquad (3)$$

There is no constant of integration as x and y vanish together.

Also, $y=d$, when $x=r$

$$\therefore -E.I.\ (d-r.\tan a)=-P.\frac{l-r}{l}.\frac{r^3}{3} \qquad (4)$$

In the same manner, if A were taken as the origin, and AB treated as above, equations similar to (1), (2), (3), and (4), would be obtained, and may be at once written down by substituting in these equations, $\pi-a$ for a, $P.\frac{r}{l}$ for $P.\frac{-r}{l}$, $l-r$ for r, and r for $l-r$.

Thus, the equation corresponding to (4) is,

$$-E.I.\left(d-\overline{l-r}.\ \tan\overline{\pi-a}\right)=-P.\frac{r}{l}.\frac{(l-r)^3}{3} \qquad (5)$$

Subtracting (5) from (4), $E.I.l.\ \tan a=\frac{P}{3}.r.l-r.l-2r$ (6)

And from (4), $E.I.d=\frac{P}{3}.\frac{r^2.(l-r)^2}{l}$ (7)

Equations (1), (2), (3), and (6), fully determine OB.

Next, at any point, (x, y), of BA,

$$-E.I.\frac{d^2y}{dx^2}=P.\frac{l-r}{l}.x-P.(x-r) \qquad (8)$$

Integrating, $-E.I.\frac{dy}{dx}=P.\frac{l-r}{l}.\frac{x^2}{2}-P.\left(\frac{x^2}{2}-r.x\right)+c_3$, c_3 being a constant of integration.

But $\frac{dy}{dx}=\tan a$, when $x=r$, and $\therefore\ -E.I.\tan a=P.\frac{l-r}{l}.\frac{r^2}{2}+P.\frac{r^2}{2}+c_3$.

Hence, $-E.I.\left(\frac{dy}{dx}-\tan a\right)=P.\frac{l-r}{l}.\left(\frac{x^2}{2}-\frac{r^2}{2}\right)-\frac{P}{2}.(x-r)^2$ (9)

Integrating, $-E.I.(y-x.\tan a)=P.\frac{l-r}{l}.\left(\frac{x^3}{6}-\frac{r^2}{2}.x\right)$

$$-\frac{P}{2}.\left(\frac{x^3}{3}-x^2.r+r^2.x\right)+c_4,$$ c_4 being a constant of integration.

But $y=d$, when $x=r$, and $-E.I.(d-r.\tan a)=-\frac{P}{3}.\frac{l-r}{l}.r^3$ by (4)

$$\therefore -\frac{P}{3}.\frac{l-r}{l}.r^3=-P.\frac{l-r}{l}.\frac{r^3}{3}-\frac{P}{6}.r^3+c_4, \text{ or } c_4=\frac{P}{6}.r^3$$

Hence, $-E.I.(y-x.\tan a)=P.\frac{l-r}{l}\left(\frac{x^3}{6}-\frac{r^2}{2}.x\right)-\frac{P}{6}.(x-r)^3$. (10)

Equations (8), (9), (10), and (6), fully determine BA.

Cor.—The deflection is a maximum when $\frac{dy}{dx}=0$, i.e., from (9), when

$$P.\frac{l-r}{l}.\left(\frac{x^2}{2}-\frac{r^2}{2}\right)-\frac{P}{2}.(x-r)^2=E.I.\ \tan a=\frac{P}{3}.\frac{r}{l}.l-r.l-2.r,$$

$$\text{and } \therefore\ x=l-\sqrt{\frac{l^2-r^2}{3}}.$$

This gives the position of the most deflected point, and the corresponding value of y may be found from (10).

Ex.–P=20,000-lbs., l=90-ft., r=10-ft.

$$\therefore E.I.\ \tan a=\frac{20,000}{3}.\frac{10}{90}.80.70=\frac{112,000,000}{27}$$

$$\text{and } E.I.d=\frac{20,000}{3}.\frac{10^2.80^2}{90}=\frac{1,280,000,000}{27}$$

The distance of the most deflected point from $A=\sqrt{\frac{90^2-10^2}{3}}$ =51.64-ft.

" " " " " O =38.36-ft.

The maximum deflection is the value of y obtained by putting x = 38.36-ft. in (10).

(6).–*To discuss the form of the neutral axis of a girder OA, which rests upon supports at O and A and carries several weights* P_1, P_2, P_3, ...*at points* 1, 2, 3,......*of which the distances from O are* r_1, r_2, r_3, ...*respectively.*–

It may be assumed that the total effect of all the weights is the sum of the effects of the separate weights, and thus each may be treated independently.

Let a_1, a_2, a_3,......be the slopes at the points 1, 2, 3......of the neutral axis, respectively.

Considering P_1, the equation to $O1$ is

$$-E.I.(y-x\cdot\tan a_1)=P_1.\frac{l-r_1}{l}.\left(\frac{x^3}{6}-\frac{r_1^2}{2}.x\right)$$

Considering P_1, the equation to $1A$ is

$$-E.I.(y - x.\tan a_1) = P_1.\frac{l-r_1}{l}\cdot\left(\frac{x^3}{6} - \frac{r_1^2}{2}.x\right) - \frac{P_1}{6}.(x \quad r_1)^3$$

" " P_2, the equation to $O2$ is

$$-E.I.(y - x.\tan a_2) = P_2.\frac{l-r_2}{l}\cdot\left(\frac{c^3}{6} - \frac{r_2^2}{2}.x\right)$$

" " P_2, the equation to $2A$ is

$$-E.I.(y - x.\tan a_2) = P_2.\frac{l-r_2}{l}\cdot\left(\frac{x^3}{6} - \frac{r_2^2}{2}.x\right) - \frac{P_2}{6}.(x - r_2)^3$$

and so on for P_3, P_4,.........

The *total* deflection at any point is the sum of the deflections due to the several loads.

Thus, the deflection at any point, (x,y), between 3 and 4 = deflection due to P_1 + that due to P_2 +, &c.,

Now, def.n due to $P_1 = x.\tan a_1 - \frac{P_1}{E.I}\cdot\frac{l-r_1}{l}\left(\frac{x^3}{6} - \frac{r_1^2}{2}x\right) + \frac{1}{6}\cdot\frac{P_1}{E.I}\cdot(x - r_1)^3$

" " " $P_2 = x.\tan a_2 - \frac{P_2}{E.I}\cdot\frac{l-r_2}{l}\cdot\left(\frac{x^3}{6} - \frac{r_2^2}{2}.x\right) + \frac{1}{6}\cdot\frac{P_2}{E.I}(x - r_2)^3$

" " " $P_3 = x.\tan a_3 - \frac{P_3}{E.I}\cdot\frac{l-r_3}{l}\cdot\left(\frac{x^3}{6} - \frac{r_3^2}{2}.x\right) + \frac{1}{6}\cdot\frac{P_3}{E.I}\cdot(x - r_3)^3$

" " " $P_4 = x.\tan a_4 - \frac{P_4}{E.I}\cdot\frac{l-r_4}{l}\cdot\left(\frac{x^3}{6} - \frac{r_4^2.x}{2}\right)$

and so on.

Hence, the total deflection $= Y =$

$$x.(\tan a_1 + \tan a_2 + \tan a_3 + \ldots\ldots)$$
$$-\frac{1}{E.I.l}\cdot\left\{\frac{x^3}{6}\cdot\left(P_1.\overline{l-r_1} + P_2.\overline{l-r_2} + \ldots\ldots\right) - \frac{x}{2}\left(P_1.r_1^2 + P_2.r_2^2 + \ldots\ldots\right)\right\}$$
$$+\frac{1}{6.E.I}\cdot\left\{P_1.\overline{x-r_1}^3 + P_2.\overline{x-r_2}^3 + P_3.\overline{x-r_3}^3\right\}$$

Again, if the most deflected point lies between 3 and 4, its position may be at once determined by making $\frac{dy}{dx}$ zero. This gives a quadratic for finding x, and the corresponding deflection may be obtained by substituting this value of x in the above equation.

Ex.–A girder of 100-ft. span supports two weights of 20,000-lbs. and 30,000-lbs. at points distant 20-ft. and 60-ft. from one end, respectively.

The most deflected point evidently lies between the two weights.

According to the above, if there are only two weights P_1, P_2, the equation to the neutral axis between them is,

$$y = x.\tan a_1 - \frac{P_1}{E.I}.\frac{l-r}{l}\cdot\left(\frac{x^3}{6} - \frac{r_1^2}{2}.x\right) + \frac{P_1}{6.E.I}(x-r_1)^3 + x.\tan a_2$$
$$-\frac{P_2}{E.I}.\frac{l-r_2}{l}.\left(\frac{x^3}{6} - \frac{r_2^2}{2}.x\right).$$

Also, y is a maximum when $\frac{dy}{dx}=0$, i.e., when

$$0 = \tan a_1 - \frac{P_1}{E.I}.\frac{l-r_1}{l}.\left(\frac{x^2}{2} - \frac{r_1^2}{2}\right) + \frac{P_1}{2\cdot E.I}(x-r_1)^2 + \tan a_2$$
$$-\frac{P_2}{E.I}.\frac{l-r_2}{l}.\left(\frac{x^2}{2} - \frac{r_2^2}{2}\right)$$

In the present case, $P_1 = 20{,}000$-lbs., $P_2 = 30{,}000$-lbs., $r_1 = 20$-ft., $r_2 = 60$-ft., $\therefore E.I.\tan a_1 = 6{,}400{,}000$, and $E.I.\tan a_2 = -4{,}800{,}000$, and the quadratic becomes, $x^2 + 100\cdot x - 7600 = 0$, or $x = 50\cdot497$-ft. $\therefore$ &c.

(7).—*Girder encastré at the ends.*—The girder $BCDEFG$ rests upon supports at the ends, is held in position by blocks forced between the ends and the abutments, and carries a uniformly distributed load of intensity w.

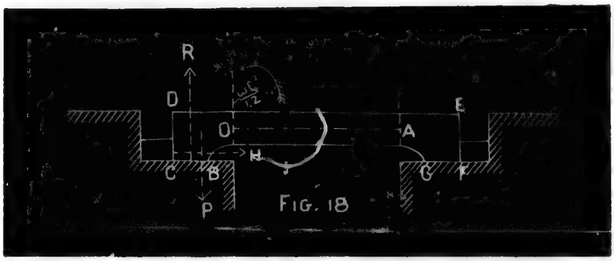


FIG. 18

It is required to determine the pressure that must be developed between the blocks and the girder, so that the straight portion between vertical sections at points O and A of the neutral axis may be in the same condition as if the girder were *fixed* at these sections.

Let l be the length of OA.

Let R be the reaction at the surface BC, and r its distance from O.

Let H be the reaction between the block and the end CD, and h its distance from O.

Let P be the weight of the segment on the left of the vertical section O, and p its distance from O.

The shearing force at O is evidently $\frac{w.l}{2}$, and it can easily be proved, as in Ex. 7, § 4, that the moment of the couple, *due to fixture* at O,

is $\frac{w.l^2}{12}$; $\therefore$ for the equilibrium of the segment on the left of the section at O,

$$R+\frac{w.l}{2}-P=0, \text{ and } R.r-P.p-H.h-\frac{w.l^2}{12}=0.$$

$$\therefore R=P-\frac{w.l}{2}, \text{ and } H=\frac{\left(P-\frac{w.l}{2}\right)\cdot r-P.p-\frac{w.l^2}{12}}{h}$$

= the required pressure.

Again, take O as the origin, OA as the axis of x, and a vertical through O as the axis of y,

$\therefore$ at any point $(x.y)$ of the neutral axis,

$$-E.I.\frac{d^2y}{dx^2}=\frac{w.l}{2}.x-\frac{w.x^2}{2}-\frac{w.l^2}{12}.$$

(8).—*On the stiffness of a beam.*—The stiffness is measured by the ratio of the load to the maximum deflection due to such load.

Ex. 1.—A girder fixed at one end and loaded with a weight W at the other.

$$\therefore D=\text{the maximum deflection}=\frac{1}{3}\frac{W.l^3}{E.I}, \text{ and } \frac{W}{D}=3.E.\frac{I}{l^3}.$$

Ex. 2.—A girder fixed at one end and loaded uniformly with a weight of intensity w.

$$\therefore D=\tfrac{1}{8}.\frac{w.l^4}{EI}, \text{ and } \frac{W}{D}=\frac{w.l}{D}=8.E.\frac{I}{l^3}.$$

Ex. 3.—A girder resting upon two supports and carrying a weight W at the centre.

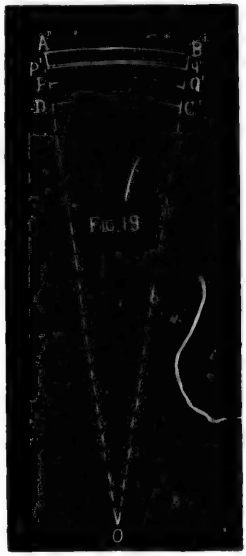
$$\therefore D=\frac{1}{48}.\frac{W.l^3}{E.I}. \text{ and } \frac{W}{D}.=48.\ E.\frac{I}{l^3}.$$

Ex. 4.—A girder resting upon two supports and carrying a uniformly distributed load of intensity w.

$$\therefore D=\frac{5}{384}.\frac{w.l^4}{E.I}. \text{ and } \frac{W}{D}=\frac{w.l}{D}=\frac{384}{5}.E.\frac{I}{l^3}.$$

In all these cases the stiffness varies directly as the moment of inertia, and inversely as the cube of the length.

If the girder is rectangular in section and of constant breadth, then I varies directly as the cube of the depth (d), and therefore the stiffness $\propto \left(\frac{d}{l}\right)^3$. This shews how advantageous it is, in point of stiffness, to increase the depth in proportion to the length.



(9).—*On the work done in bending a beam.* Let $A'B'C'D'$ be an originally rectangular element of a beam strained under the action of external forces.

Let the surfaces $A'D'$, $B'C'$, meet in O; O is the centre of curvature of the arc $P'Q'$ of the neutral axis.

Let $OP'=R=OQ'$.

Let the length of the arc $P'Q'=dx$.

Consider any elementary fibre $p'q'$, of length dx', of sectional area a, and distant y from the neutral axis.

Let t be the stress in $p'q'$.

The work done in stretching $p'q'$

$$=\frac{t}{2}\cdot\left(dx'-dx\right)\cdot$$

But $\dfrac{dx'}{dx}=\dfrac{p'q'}{P'Q'}=\dfrac{R+y}{R}$, and $t=E.a\dfrac{dx'-dx}{dx}=E.a.\dfrac{y}{R}$.

$$\therefore \text{ the work done in stretching } p'q'=\frac{1}{2}\cdot\frac{E}{R^2}\cdot dx.a.y^2,$$

and the work done in deforming the prism $A'B'C'D'=\Sigma\left(\dfrac{1}{2}\cdot\dfrac{E}{R^2}\cdot dx.\,a.y^2\right)$

$$=\frac{1}{2}\cdot\frac{E}{R^2}\cdot dx.\Sigma\,(a.y^2)=\frac{1}{2}\cdot\frac{E.I}{R^2}\cdot dx$$

Hence, the total work between two sections of abscissæ x_1,x_2,

$$=\int_{x_1}^{x_2}\frac{1}{2}\cdot\frac{E.I}{R^2}\cdot dx=\frac{E.I}{2}\int_{x_1}^{x_2}\frac{dx}{R^2}.$$

But, $\dfrac{1}{R}=\dfrac{M}{E.I}$; $\therefore$ the work between the given limits

$$=\frac{E.I.}{2}\int_{x_1}^{x_2}dx\left(\frac{M}{E.I}\right)^2=\frac{1}{2.E.I.}\int_{x_1}^{x_2}M^2.\,dx.$$

This expression is necessarily equal to the work of the external forces between the same limits, and is also the semi vis-viva acquired by the beam in changing from its natural state of equilibrium.

Cor.–If the proof load P is concentrated at one point of a beam, and if d is the proof-deflection, $\therefore$ the *resilience* $=\frac{P}{2}.d$.

If a proof load of intensity w is uniformly distributed over the beam. and if y is the deflection at any point, $\therefore$ the resilience $=\frac{1}{2}\int w.y.dx$, the integration extending throughout the whole length of the beam.

The case of the single weight, however, is the most useful in practice.

(10).–*On the transverse vibrations of a beam resting upon two supports in the same horizontal plane.*

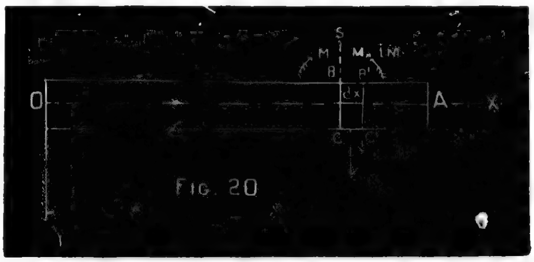
It is assumed :—

(*a*).–That the beam is homogeneous and of uniform sectional area.

(*b*).–That the axis (*neutral*) remains unaltered in length.

(*c*).–That the vibrations are small.

(*d*).–That the particles of the beam vibrate in the vertical planes in which they are primarily situated. In reality, these particles have a slight angular motion about the horizontal axis through the centre of gravity of the section, but for the sake of simplicity the effect of this motion is disregarded.



Let OA be the beam.

Take O as the origin, the neutral line OA as the axis of x, and the vertical OY as the axis of y.

Consider an element of the beam, bounded by the vertical planes BC, $B'C'$, of which the abscissæ are, x and $x+dx$, respectively.

Let w be the intensity of the load per unit of length, $\therefore$ $w.dx$ is the load upon the given element, and acts vertically through its centre.

Let S be the shearing force at B, $S+dS$ is the shearing force at B'.

Let M be the bending moment at B, $M+dM$ is the bending moment at B'.

Also, the resistance of the element to acceleration $=\frac{w}{g}.\frac{d^2y}{dt^2}$.

Hence, at any time t,

$$\frac{w}{g}\cdot dx.\frac{d^2y}{dt^2}+S-(S+dS)-w.dx=0$$

$$\text{or, }\frac{d^2y}{dt^2}-\frac{g}{w}\cdot\frac{dS}{dx}-g=0. \qquad (1)$$

Again, taking moments about the middle point of BB' or CC',

$$\therefore\ M-(M+dM)+S.\frac{dx}{2}+(S+dS).\frac{dx}{2}=0,$$

$$\text{or }\frac{dM}{dx}=S. \qquad (2)$$

But $M=-E.I.\frac{d^2y}{dx^2}$, $\therefore\ S=-E.I.\frac{d^3y}{dx^3}$, and $\frac{dS}{dx}=-E.I.\frac{d^4y}{dx^4}$.

$$\text{Hence from (1), }\frac{d^2y}{dt^2}+\frac{g}{w}\cdot E.I.\frac{d^4y}{dx^4}-g=0. \qquad (3)$$

This equation does not admit of a finite integration, but may be integrated in the form of a partial differential equation.

(11).–*Continuous girders.* When a girder overhangs its bearings, or is supported at more than two points, it assumes a wavy form, and is said to be *continuous.* The convex portions are in the same condition as a loaded girder resting upon a single support, the upper layers of the girder being extended and the lower compressed. The concave portions are in the same condition as a loaded girder supported at two points, the upper layers being compressed and the lower extended. At certain points called *points of contrary flexure,* or *points of inflexion,* the curvature changes sign and the flange stresses are necessarily zero. Hence, apart from other practical considerations, the flanges might be wholly severed at these points without endangering the stability of the girder.

(12).–*To discuss the form $OPXQV$ of a continuous girder resting upon three supports at O, X and V, when the segment OX is loaded with a weight w_1 per unit of length, and the segment XV with a weight w_2 per unit of length.*

Let $OX=l_1$, $XV=l_2$.

Let a be the angle made with OV by the tangent at X common to the segments XPO and XQV.

Let R_1, R_2, R_3, be the reactions at O, X, V, respectively.

Consider the segment OPX, and refer it to rectangular axes OX, OY.

The equation of moments at any point $P_1(x,y)$, is :-

$$-E.I.\frac{d^2y}{dx^2}=R_1.x-\frac{w_1.x^2}{2}=M \qquad 1$$

Integrating, $E.I.\frac{dy}{dx}=R_1\frac{x^2}{2}-\frac{w_1.x^3}{6}+c$, c being a constant of integration.

But $\frac{dy}{dx}=tan\ a$, when $x=l_1$,

$$\therefore -E.I.\ tan\ a=R_1.\frac{l_1^2}{2}-w_1.\frac{l_1^3}{6}-c$$

Hence, $-E.I.\left(\frac{dy}{dx}-tan\ a\right)=R_1.\left(\frac{x^2}{2}-\frac{l_1^2}{2}\right)-w_1.\left(\frac{x^3}{6}-\frac{l_1^3}{6}\right) \qquad 2$

Integrating, $-E.I.(y-x.tan\ a)=R_1.\left(\frac{x^3}{6}-\frac{l_1^2}{2}.x\right)-w_1.\left(\frac{x^4}{24}-\frac{l_1^3}{6}.x\right) \qquad 3$

There is no constant of integration in this equation as x and y vanish together.

Also $y=0$, when $x=l_1$,

$$\therefore\text{, from 3, } E.I.\ tan\ a=-\frac{R_1}{3}.l_1^2+\frac{w_1}{8}.l_1^3 \qquad 4$$

Similarly, the segment VX gives,

$$E.I.\ tan\,(\pi-a)=-\frac{R_2}{3}\,l_2^2+\frac{w_2}{8}.l_2^3 \qquad 5$$

$$\therefore\text{ from (4) and (5), } R_1.l_1^2+R_2.l_2^2=\tfrac{3}{8}.w_1.l^3+\tfrac{3}{8}w_2.l_2^3 \qquad 6$$

Again, by taking moments about X,

$$R_1.l_1-R_2.l_2=\tfrac{1}{2}.w_1.l_1^2-\tfrac{1}{2}.w_2.l_2^2. \qquad 7$$

Hence from (6) and (7),

$$R_1=\frac{3.w_1.l^3+4.w_1l_1^2\,l_2-w_2.l_2^3}{8.l_1.(l_1+l_2),} \qquad 8$$

$$R_2=\frac{-w_1.l_1^3+4.w_2.l_1.l_2^2+3w_2.l_2^3}{8.l_2.(l_1+l_2)} \qquad 9$$

$$\therefore R_2 = w_1 l_1 + w_2 l_2 - R_1 - R_3$$

$$= \frac{5.l_1.l_2.(w_1.l_1^3 + w_2.l_2^3) + 4.l_1^2.l_2^2\,(w_1 + w_2) + w_1.l_1^4 + w_2.l_2^4}{8.l_1.l_2.(l_1 + l_2)} \qquad 10$$

From (4) and (8), $E.I.tan\ a = \frac{l_1.l_2(w_2.l_2^2 - w_1.l_1^2)}{24.(l_1 + l_2)}$ 11

At the *points of inflexion*, the flange stresses, and therefore the *bending* moments, are zero.

Hence, from 1, distance of point of inflexion in OX from $O = \frac{2.R_1}{w_1}$.

So " " " VX " $V = \frac{2.R_3}{w_2}$.

The deflection at any point of OX is given by 3, and the distance of the most deflected point is the value of x obtained by putting $\frac{dy}{dx} = 0$ in 2.

The point at which the *bending moment* is a maximum is found by putting $\frac{dy}{dx} = 0$, or is midway between the end of the girder, and the nearest point of inflection.

Ex.–Let $l_1 = l_2 = l$, $\therefore R_1 = \frac{l}{16}.(7.w_1 - w_2)$, $R_2 = \frac{5.l}{8}.(w_1 + w_2)$,

$$R_3 = \frac{l}{16}.(-w_1 + 7w_2), \text{ and } E.I.tan\ a = \frac{l^3}{48}.(w_2 - w_1).$$

Also, the distance from O of the point in OX at which the bending moment is a maximum $= \frac{R_1}{w_1}$.

$\therefore$ from (1), the maximum bending moment in $OX = \frac{1}{2}.\frac{R_1^2}{w_1}$.

So, " " " ... $VX = \frac{1}{2}.\frac{R_3^2}{w_2}$.

(13).–*To discuss the form $OAXBV$ of a continuous girder resting upon three supports at O, X, and V, when a weight W_1 is suspended from a point A distant r_1 from O, and a weight W_2 from a point B, distant R_2 from V.*

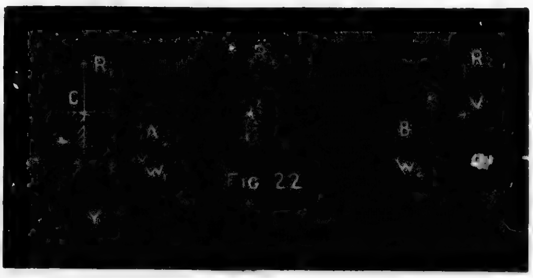


FIG 22

Take the same notation as before, and let β_1, β_2, be the angles which the tangents to the girder at A and B, respectively, make with OV.

The equations determining OA (§ 7) are:—

$$-E.I.\frac{d^2y}{dx^2} = R_1.x \tag{1}$$

$$-E.I.\left(\frac{dy}{dx} - \tan\beta_1\right) = R_1.\left(\frac{x^2}{2} - \frac{r_1^2}{2}\right) \tag{2}$$

$$-E.I.\left(y - x.\tan\beta_1\right) = R_1.\left(\frac{x_3}{6} - \frac{r_1^2}{2}.x\right) \tag{3}$$

The equations determining AX are:—

$$-E.I.\frac{d^2y}{dx^2} = R_1.x - W_1.(x - r_1) \tag{4}$$

$$-E.I.\left(\frac{dy}{dx} - \tan\beta_1\right) = R_1.\left(\frac{x^2}{2} - \frac{r_1^2}{2}\right) - \frac{W_1}{2}.(x - r_1)^2 \tag{5}$$

$$-E.I.\left(y - x.\tan\beta_1\right) = R_1.\left(\frac{x^3}{6} - \frac{r_1^2}{2}.x\right) - \frac{W_1}{6}.(x - r_1)^3 \tag{6}$$

But at X, $\frac{dy}{dx} = \tan a$, $x = l_1$, and $y = 0$.

$\therefore$ from (5) and (6), $-E.I.(\tan a - \tan\beta_1) = R_1.\left(\frac{l_1^2}{2} - \frac{r_1^2}{2}\right) - \frac{W_1}{2}.(l_1 - r_1)^2$

$$\text{and } E.I.l_1.\tan\beta_1 = R_1.\left(\frac{l_1^3}{6} - \frac{r_1^2}{2}.l_1\right) - \frac{W_1}{6}.(l_1 - r_1)^3$$

$$\text{Hence, } -E.I.\tan a = \frac{R_1}{3}.l_1^2 - \frac{W_1}{6}.(l_1 - r_1)^2.\frac{2l_1 + r_1}{l_1} \tag{7}$$

Similarly the segment VX gives,

$$-E.I.\tan(\pi - a) = \frac{R_3}{3}.l_2^2 - \frac{W_2}{6}.(l_2 - r_2)^2.\frac{2.l_2 + r_2}{l_2} \tag{8}$$

$\therefore$ from (7) and (8),

$$R_1.l_1^2 + R_3.l_2^2 = \frac{W_1}{2}.(l_1 - r_1)^2.\frac{2.l_1 + r_1}{l_1} + \frac{W_2}{2}.(l_2 - r_2)^2.\frac{2.l_2 + r_2}{l_2} \tag{9}$$

Again, by taking moments about X,

$$R_1.l_1 - R_3.l_2 = W_1.(l_1 - r_1) - W_2.(l_2 - r_2) \tag{10}$$

Hence from (9) and (10),

$$R_1 = \frac{1}{l_1.(l_1 + l_2)}.\left\{ .W_1.(l_1 - r_1)\left(\frac{[l_1 - r_1][2.l_1 + r_1]}{2.l_1} + l_2\right) - W_2.\frac{r_2}{2.l_2}.(l_2^2 - r_2^2) \right\}$$

$$R_3 = \frac{1}{l_2.(l_1 + l_2)}.\left\{ -W_1.\frac{r_1}{2.l_1}.(l_1^2 - r_1^2) + W_2.(l_2 - r_2).\left(\frac{[l_2 - r_2]\,[2.l_2 + r_2]}{2.l_2} + l\right) \right\}$$

and $R_2 = W_1 + W_2 - R_1 - R_3 = \text{\&c.}$

(14).–*Swing-Bridges.*–The two preceding theorems are of importance in considering the equilibrium of swing-bridges which revolve about a single support at the pivot pier.

From § (12), if $w_2 = 0$, $\therefore R_3 = -\frac{1}{8} \cdot \frac{w_1 . l_1^3}{l_2 . (l_1 + l_2)}$

From § (13), if $W_2 = 0$, $\therefore R_3 = -\frac{1}{2} \cdot \frac{W_1 . r_1 . (l_1^2 - r_1^2)}{l_1 . l_2 . (l_1 + l_2)}$

Hence, if the segment XV is unloaded, R_3 is *negative*, so that the end V will leave its support, and *hammering* will ensue. This evil is usually obviated by one of the following methods:—

(*a*).–When the girder merely rests upon the supports, the segment XV may be loaded in such a manner as to make R_3 *zero* or *positive*. This result is attained if $w_1 . l_1^3 < 4 . w_2 . l_1 . l_2^2 + 3 . w_2 . l_2^3$, § (12),

or if $W_1 . \frac{r_1}{2 . l_1} . (l_1^2 - r_1^2) < W_2 . (l_2 - r_2) . \left(\frac{(l_2 - r_2)(2 . l_2 + r_2)}{2 l_2} + l_1 \right)$, § (13).

(*b*).–The ends are sometimes prevented from rising by a latching apparatus.

(*c*).–An upward pressure, at least equal to the corresponding negative reaction, may be exerted by suitable machinery upon each end of the girder, which is thus wholly prevented from leaving its seat.

(15).–*The Theorem of Three Moments.*–It is required to determine a relation between the *Bending-Moments* at any *three consecutive* points of support of a loaded continuous girder of several spans.

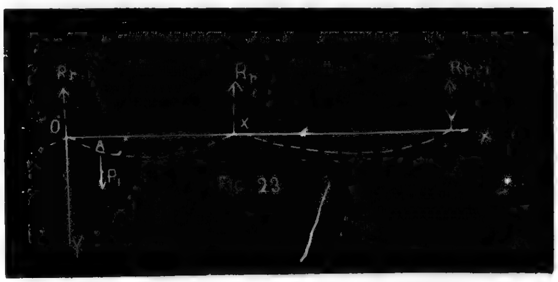


Fig. 23

Let O, X, V, be the $(r-1)^{th}$, r^{th}, and $(r+1)^{th}$ supports respectively.

Let $OX = l_r$, $XV = l_{r+1}$

Case A.–Let w_r be the load per unit of length on OX, w_{r+1} the load per unit of length on XV.

Let R_{r-1}, R_r, R_{r+1}, be the reactions at O, X, V, respectively.

Let M_{r-1}, M_r, M_{r+1}, be the bending moments at O, X, V, respectively.

Let a be the angle which the tangent to the girder at X makes with OV.

Consider the segment OX, and refer it to the rectangular axes OX, OY.

The equation of moments at any point (x, y) is,

$$-E.I.\frac{d^2y}{dx^2}=R_{r-1}.x-w_r.\frac{x^2}{2}+M_{r-1}=M \qquad (1)$$

At X, $x=l_r$ and $M=M_r$

$$\therefore R_{r-1}.l_r-w_r.\frac{l_r^2}{2}+M_{r-1}=M_r \qquad (2)$$

Similarly, the segment XV gives,

$$R_{r+1}.l_{r+}\ -w_{r+1}.\frac{l^2_{r+1}}{2}+M_{r+1}=M_r \qquad (3)$$

Combining (2) and (3),

$$R_{r-1}.l_r^2+R_{r+1}.l_{r+1}^2-w_r.\frac{l_r^3}{2}-w_{r+1}.\frac{l^3_{r+1}}{2}+M_{r-1}.l_r+M_{r+1}.l_{r+1}$$
$$=M_r.(l_r+l_{r+1}) \qquad (4)$$

Integrating (1),

(5) $-E.I.\frac{dy}{dx}=R_{r-1}.\frac{x^2}{2}-w_r.\frac{x^3}{6}+M_{r-1}.x+c$, c being a constant of integration. 5

When $x=l_r, \frac{dy}{dx}=tan\ a$,

(6) $$\therefore -E.I.\ tan\ a=R_{r-1}.\frac{l_r^2}{2}-w_r.\frac{l_r^3}{6}+M_{r-1}.l_r+c$$ 6

Hence,

$$-E.I.\left(\frac{dy}{dx}-tan\ a\right)=R_{r-1}.\left(\frac{x^2}{2}-\frac{l_r^2}{2}\right)-w_r.\left(\frac{x^3}{6}-\frac{l_r^3}{6}\right)+M_{r-1}.(x-l_r)$$

Integrating,

$$-E.I.(y-x.tan\,a)=R_{r-1}.\left(\frac{x^3}{6}-\frac{l_r^2}{2}.x\right)-w_r.\left(\frac{x^4}{24}-\frac{l_r^3}{6}.x\right)$$
$$+M_{r-1}.\left(\frac{x^2}{2}-l_r.x\right) \qquad (6)$$

There is no constant of integration as x and y vanish together.

Also, when $x=l_r$, $y=0$,

$$\therefore E.I.l_r.\ tan\ a=-R_{r-1}.\frac{l_r^3}{3}+w_r.\frac{l_r^4}{8}-M_{r-1}.\frac{l_r^2}{2}$$

$$\text{or } E.I.\ tan\ a=-R_{r-1}.\frac{l_r^2}{3}+w_r.\frac{l_r^3}{8}-M_{r-1}.\frac{l_r}{2} \qquad (7)$$

Similarly, the segment XV gives,

$$E.I.\,tan\,(\pi-a)=-R_{r+1}.\frac{l^2_{r+1}}{3}+w_{r+1}.\frac{l^3_{r+1}}{8}-M_{r+1}\frac{l_{r+1}}{2} \qquad (8)$$

Adding the two last equations and transposing,

$$R_{r-1}.l_r^2 + R_{r+1}.l_{r+1}^2 = \frac{3}{8}.w_r.l_r^3 + \frac{3}{8}.w_{r+1}.l_{r+1}^3 - \frac{3}{2}.M_{r-1}.l_r - \frac{3}{2}.M_{r+1}.l_{r+1} \quad (9)$$

Finally, combining (4) and (9),

Import.

$$M_{r-1}\,l_r + 2.M_r.(l_r + l_{r+1}) + M_{r+1}\,l_{r+1} = -\tfrac{1}{4}.(w_r.l_r^3 + w_{r+1}.l_{r+1}^3). \quad (10)$$

which is the relation required.

If the girder is supported at n points, there are $n-2$ equations connecting the corresponding bending moments, and two additional equations result from the conditions of support at the ends.

For example, if the ends merely rest on the supports, $\therefore M_1=0$ and $M_n=0$; if an end is *fixed,* $\frac{dy}{dx}=0$ at that point.

Case B.–Let the loads upon OX, XV, respectively consist of a number of weights P_1, P_2, P_3.......distant p_1, p_2, p_3.......from O, and Q_1, Q_2. Q_3......distant q_1, q_2, q_3......from V. Refer the neutral axis OAX to the rectangular axes OX, OY.

It may be assumed that the total effect of all the weights is the algebraic sum of the effects of the weights taken separately.

Consider the effect of P_1 at A.

Let a_1 be the angle which the tangent at A makes with OX.

Let a_2 be the angle which the tangent at X makes with OX.

The equation of moments at any point x, y, of the neutral axis *between O and A*, is,

$$-E.I.\frac{d^2y}{dx^2} = R_{r-1}.x - M_{r-1} \quad (1)$$

Integrating, $\therefore -E.I.\frac{dy}{dx} = R_{r-1}.\frac{x^2}{2} - M_{r-1}.x + c_1$,

c_1 being a constant of integration.

At A, $\frac{dy}{dx} = tan\ a_1$, and $x=p_1$,

$$\therefore -E.I.\ tan\ a_1 = R_{r-1}.\frac{p_1^2}{2} - M_{r-1}.p_1 + c_1$$

Subtracting :—

$$\therefore -E.I.\left(\frac{dy}{dx} - tan\ a_1\right) = R_{r-1}.\left(\frac{x^2}{2} - \frac{p_1^2}{2}\right) - M_{r-1}.\left(x - p_1\right) \quad (2)$$

Integrating :—

$$\therefore -E.I.(y - x.\ tan\ a_1) = R_{r-1}.\left(\frac{x^3}{6} - \frac{p_1^2}{2}.x\right) - M_{r-1}.\left(\frac{x^2}{2} - p_1.x\right) \quad (3)$$

There is no constant of integration as x and y vanish together.

At A, $x=p_1$, and let $y=y_p$

$$\therefore -E.I.(y_1 - p_1.\tan a_1) = -R_{r-1}.\frac{p_1^3}{3} + M_{r-1}.\frac{p_1^2}{2} \tag{4}$$

Equations 1, 2, and 3 hold for all points *between* O and A.

The equation of moments at any point x, y, *between* A and X is,

$$-E.I.\frac{d^2y}{dx^2} = R_{r-1}.x - P.(x - p_1) - M_{r-1} \tag{5}$$

Integrating:—

$$\therefore -E.I.\frac{dy}{dx} = R_{r-1}.\frac{x^2}{2} - P.\left(\frac{x^2}{2} - p_1.x\right) - M_{r-1}.x + c_2$$

c_2 being a constant of integration.

$$\therefore \text{ at } A, -E.I.\ \tan a_1 = R_{r-1}.\frac{p_1^2}{2} + P.\frac{p_1^2}{2} - M_{r-1}.p_1 + c_2$$

Subtracting:—

$$-E.I.\left(\frac{dy}{dx} - \tan a_1\right) = R_{r-1}.\left(\frac{x^2}{2} - \frac{p_1^2}{2}\right) - \frac{P_1}{2}.(x - p_1)^2 - M_{r-1}.(x - p_1) \tag{6}$$

Integrating:— $$-E.I.(y - x\tan a_1) = R_{r-1}.\left(\frac{x^3}{6} - \frac{p_1^2}{2}.x\right)$$
$$- P_1.\left(\frac{x^3}{6} - p_1.\frac{x^2}{2} + \frac{p_1^2}{2}.x\right) - M_{r-1}\left(\frac{x^2}{2} - p_1.x\right) + c_3,$$

c_3 being a constant of integration.

$$\text{At } A, -E.I.(y_1 - p_1.\tan a_1) = -R_{r-1}.\frac{p_1^3}{3} - P_1.\frac{p_1^3}{6} - M_{r-1}.\frac{p_1^2}{2} + c_4$$
$$= -R_{r-1}.\frac{p_1^3}{3} - M_{r-1}.\frac{p_1^2}{2}, \text{ by } 4$$

Hence $c_4 = P.\frac{p_1^3}{6}$, and

$$\therefore -E.I.(y - x.\tan a_1) = R_{r-1}.\left(\frac{x^3}{6} - \frac{p_1^2}{2}.x\right) - \frac{P_1}{6}.(x - p_1)^3$$
$$- M_{r-1}.\left(\frac{x^2}{2} - p_1.x\right) \tag{7}$$

At X, $\frac{dy}{dx} = \tan a_2, x = l_r$, and $y = 0$,

$$\therefore \text{ by } 6, -E.I.(\tan a_2 - \tan a_1) = R_{r-1}.\left(\frac{l_r^2}{2} - \frac{p_1^2}{2}\right)$$
$$- \frac{P_1}{2}.(l_r - p_1)^2 - M_{r-1}.(l_r - p_1) \tag{8}$$

$$\text{and by } 7, +E.I.l_r.\tan a_1 = R_{r-1}.\left(\frac{l_r^3}{6} - \frac{p_1^2}{2}.l_r\right)$$
$$- \frac{P_1}{6}.(l_r - p_1)^3 - M_{r-1}\left(\frac{l_r^2}{2} - p_1.l_r\right) \tag{9}$$

Eliminating $tan\ a_1$ between 8 and 9,

$$\therefore E.I.\ tan\ a_2 = -\frac{R_{r-1}}{3}.l_r^2 + \frac{P_1}{6}\cdot\left(2.l_r^2 - 3.l_r.p_1 + \frac{p_1^3}{l_r}\right) + M_{r-1}\cdot\frac{l_r}{2} \quad (10)$$

Similarly, from the span XV,

$$E.I.tan\ (\pi - a_2) = -\frac{R_{r+1}}{3}.l_{r+1}^2 + M_{r+1}\cdot\frac{l_{r+1}}{2} \quad (11)$$

From 10 and 11,

$$R_{r-1}.l_r^2 + R_{r+1}.l_{r+1}^2 = \frac{P_1}{2}\cdot\left(2.l_r^2 - 3.l_r.p_1 + \frac{p_1^3}{l_r}\right) + \frac{3}{2}M_{r-1}.l_r + \frac{3}{2}M_{r+1}.l_{r+1} \quad (12)$$

Take moments about X,

$$\therefore R_{r-1}.l_r - P_1.(l_r - p_1) - M_{r-1} = M_r = R_{r+1}.l_{r+1} - M_{r+1} \quad (13)$$

whence,

$$R_{r-1}.l_r^2 + R_{r+1}.l_{r+1}^2 = M_r.\ (l_r + l_{r+1}) + P_1.(l_r^2 - p_1.l_r) + M_{r-1}l_r + M_{r+1}l_{r+1} \quad (14)$$

$$\therefore \text{by 12 and 14}, M_{r-1}.l_r - 2.M_r.(l_r + l_{r+1}) + M_{r+1}l_{r+1} = +\frac{P_1.p_1}{l_r}.(l_r^2 - p_1^2) \quad (15)$$

The effect of each weight may be discussed in the same manner, hence the relation between M_{r-1}, M_r, and M_{r+1}, may be expressed in the form,

$$M_{r-1}.l_r - 2.M_r\ (l_r + l_{r+1}) + M_{r+1}.l_{r+1} = +\Sigma\frac{P.p}{l_r}.(l_r^2 - p^2) + \Sigma\frac{Q.q}{l_{r+1}}.(l_{r+1}^2 - q^2)$$

Cor.–The relation between M_{r-1}, M_r, M_{r+1}, for a uniformly distributed load may be at once deduced.

e. g., let a uniformly distributed load of intensity w_r cover a length $2a$ ($<l_r$) of the span OX, and let z be the distance of its centre from O.

$$\therefore \Sigma\frac{P.p}{l}\cdot(l_r^2 - p^2) = \int_{z-a}^{z+a}\frac{w_r.dp}{l_r}p.(l_r^2 - p^2) = \frac{w_r.2.a.\ z}{l_r}\cdot(l_r^2 - z^2 - a^2)$$

which reduces to $\frac{w_r l_r^3}{4}$ when $z = a = \frac{l_r}{2}$.

The same result may be obtained without the calculus, as follows :—

Let the load upon OX consist of $n+1$ equidistant weights, each equal to P.

Let b be the distance between two consecutive weights.

Let c be the distance from O to the nearest weight.

$$\therefore \Sigma\frac{P.p}{l_r}\cdot(l_r^2 - p^2) \text{ becomes,}$$

$$\frac{P}{l_r}\cdot\left\{l_r^2.c - c^3 + l_r^2.\ (c+b) - (c+b)^3 + l_r^2.(c+2.b) - (c+2.b)^3 + \ldots\right.$$
$$\left. + l_r^2.(c+n.b) - (c+n.b)^3\right\}$$

$$=\frac{P}{l_r}\cdot\Big\{ l_r^2.\overline{n+1}.c - \overline{n+1}.c^3 + l_r^2.\ b.\ (1+2+\dots+n)$$

$$-3.c^2.b.(1+2+\dots+n) - 3.c.b^2.(1^2+2^2+\dots+n^2) - b^3.\ (1^3+2^3\dots+n^3)\Big\}$$

$$=\frac{P}{l}\cdot\Big\{ (l_r^2.c. - c^3).\overline{n+1} + (l^2.\ b - 3.c^2.b).\frac{n.\overline{n+1}}{2} - 3.c.b^2.\frac{n.\overline{n+1}.2n+1}{6}$$

$$-b^3.\frac{(n.\overline{n+1})^2}{4}\Big\}$$

Now let the number of the weights increase indefinitely, and the distance between them, b, indefinitely diminish.

Also put $\overline{n+1}.P=2.a.w.$, and $2a=n.b$.

The above expression becomes,

$$\frac{2.a.w}{l_r}\cdot\Big\{ l_r^2.c - c^3 + (l_r^2 - 3.c^2).a - c.2.\ a^2.\frac{2n+1}{n} - 2.a^3\frac{n+1}{n}\Big\}$$

When $n=\infty$ this reduces to,

$$\frac{2.a.w_r}{l_r}\cdot\Big\{ l_r^2.c - c^3 + (l_r^2 - 3.c^2).a - 4.a^2.c - 2.a^3\Big\}$$

Put $z - a = c$ and the last expression becomes,

$$\frac{2\,a\,w_r}{l_r}\cdot\Big\{ l_r^2 z - z^3 - a^2.z\Big\}$$

(15).—*Swing-Bridges.*—To apply the Theorem of Three Moments to a continuous girder supported at *four* points.

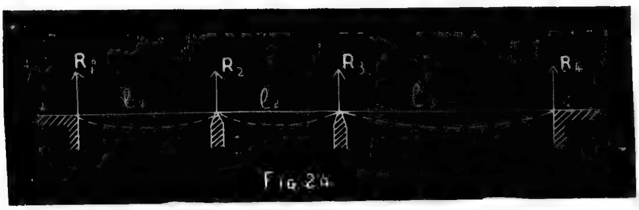


FIG. 24

Let l_1, l_2, l_3 be the lengths of the segments, w_1, w_2, w_3, the corresponding intensities of the uniformly distributed loads.

Let R_1, R_2, R_3, R_4, be the reactions at the supports; M_1, M_2, M_3, M_4, the corresponding bending moments.

$$\therefore\ M_1.l_1 + 2.M_2.(l_1+l_2) + M_3.l_2 = -\tfrac{1}{4}.(w_1.l_1^3 + w_2.l_2^3) \qquad (1)$$

$$M_2.l_2 + 2.M_3.(l_2+l_3) + M_4.l_3 = -\tfrac{1}{4}.(w_2.l_2^3 + w_3.l_3^3) \qquad (2)$$

$$\text{Also, } R_1.l_1 = \frac{w_1.l_1^2}{2} + M_2 + M_1 \qquad (3)$$

$$R_2.l_2 = l_2.\left(\frac{w_1.l_1}{2} + \frac{w_2.l_2}{2}\right) - \frac{l_2}{l_1}.(M_1+M_2) - (M_2 - M_3) \qquad (4)$$

$$R_3.l_3 = l_3.\left(\frac{w_2.l_2}{2} + \frac{w_3\ l_3}{2}\right) + \frac{l_3}{l_2}.(M_2 - M_3) - (M_3 + M_4) \qquad (5)$$

$$R_4.l = \frac{w_3.l_3^2}{2} + M_3 + M_4 \qquad (6)$$

Important deductions in reference to swing-bridges with two points of support at the pivot pier, may be made from these equations.

Let the ends of the girders rest upon the supports, and assume, as is usually the case in practice, that $w_2 = 0$.

$$\because\ M_1 = 0 \text{ and } M_4 = 0.$$

From (1) and (2), $2.M_2.(l_1 + l_2) + M_3.l_2 = -\frac{1}{4}.w_1.l_1^3,$ (7)

and $M_2.l_2 + 2.M_3.(l_2 + l_3) = -\frac{1}{4}.w_3.l_3^3.$ (8)

Hence, $$M_2 = \frac{-2.w_1.\ l_1^3.(l_2 + l_3) + w_3.l_3^3.l_2}{4.(4.l_1.l_2 + 3.l_2^2 + 4.l_1.l_3 + 4.l_2.l_3)} \qquad (9)$$

and $$M_3 = \frac{w_1.l_1^3.l_2 - 2w_3.l_3^3.(l_1 + l_2)}{4.(4.l_1l_2 + 3.l_2^2 + 4l_1.l_3 + 4l_2.l_3)}. \qquad (10)$$

From (3), $$R_1.l_1 = \frac{w_1.l_1^2}{2} + M_2$$

$$= \frac{w_1.(6.l_1^3.l_2 + 6.l_1^3.l_3 + 6.l_1^2.l_2^2 + 8.l_1^2.l_2.l_3) + w_3.l_3^3.l_2}{4.(4.l_1.l_2 + 3.l_2^2 + 4.l_1.l_3 + 4.l_2.l_3)} \qquad (11)$$

From (6,) $$R_4.l_3 = \frac{w_3.l_3^2}{2} + M_3$$

$$= \frac{w_3.(6.l_3^3.l_2 + 6.l_3^3.l_1 + 6.l_3^2.l_2^2 + 8.l_3^2.l_1.l_2) + w_1.l_1^3.l_2}{4.(4.l_1.l_2 + 3\ l_2^2 + 4.l_1.l_3 + 4.l_2.l_3)} \qquad (12)$$

Thus . R and R_4 are both *positive* for any distribution whatever of the load over the side segments, and no hammering of the ends can ever take place.

Again, if $w_3 = 0$, M is *negative*, and M_3 *positive*.

if $w_1 = 0$, M_2 is *positive* and M_3 *negative*.

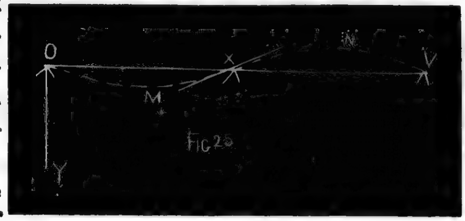
(16).–Four methods may be followed in the erection of a continuous girder, viz:—

1.–It may be built on the ground and *lifted* into place.

2.–It may be built on the ground, and rolled endwise over the piers.

3.–It may be built in position on a scaffold.

4.–Each span may be erected separately, and continuity produced by securely jointing consecutive ends, having drawn together the upper flanges. A more effective distribution of the material is often made by eaving a little space between the flanges and forming a wedge-shaped joint.



Example.—OMX and XNV are consecutive segments of a continuous girder resting upon an indefinite number of equidistant piers.

The *dead* weight of the girder is w_1 per unit of length, and each alternate (concave) segment is subjected to a rolling load of w_2 per unit of length.

Suppose that each segment is erected separately, and let M_1 be the moment of flexure which must be introduced at the points of support to ensure continuity.

Consider the segment OMX, and refer it to rectangular axes OX, OY.

Let l be the length of OX, and a_1 the slope at X.

The equation of moments at any point $(x, y,)$ is,

$$-E.I.\frac{d^2y}{dx^2} = \frac{w_1+w_2}{2}.\,l.x - \frac{w_1+w_2}{2}.x^2 - M_1 = \frac{w_1+w_2}{2}(l.x - x^2) - M_1 = M \quad (1)$$

Integrating, $-E.I.\frac{dy}{dx} = \frac{w_1+w_2}{2}\left(\frac{l.x^2}{2} - \frac{x^3}{3}\right) - M_1.\,x + c$, c being a constant of integration.

But $\frac{dy}{dx}$ is zero when $x = \frac{l}{2}$, and $\therefore\ 0 = \frac{w_1+w_2}{2}.\frac{l^3}{12} - M_1.\frac{l}{2} + c$

Hence, $-E.I.\frac{dy}{dx} = \frac{w_1+w_2}{2}.\left(\frac{l.x^2}{2} - \frac{x^3}{3} - \frac{l^3}{12}\right) - M_1.\left(x - \frac{l}{2}\right).$

Also, when $x = l, \frac{dy}{dx} = tan\ (\pi - a_1)$.

$$\therefore -E.I.tan(\pi - a_1) = \frac{w_1+w_2}{24}.\,l^3 - M_1.\frac{l}{2} \quad (2)$$

Similarly the segment VX gives,

$$-E.I.tan\,a_1 = \frac{w_1.\,l^3}{24} - M_1.\frac{l}{2} \quad (3)$$

Adding (2) and (3), $0 = \frac{2.w_1+w_2}{24}.l^3 - M_1.l$

$$\text{or } M_1 = \frac{2.w_1+w_2}{24}.l^2 \quad (4)$$

$$\text{and from (2) or (3), } E.I.\ tan\,a_1 = \frac{w_2.l^3}{48} \quad (5)$$

Again, from (1), the bending moment (M') at the centre of OX

$$=\frac{w_1+w_2}{2}.\frac{l^2}{4}-\frac{2.w_1+w_2}{24}.l^2=\frac{w_1+2.w_2}{24}.l^2 \qquad (6)$$

Thus, $M_1+M'=\frac{2.w_1+w_2}{24}.l^2+\frac{w_1+2.w_2}{24}.l^2=\frac{w_1+w_2}{8}.l^2$, which is the bending moment at the centre of the girder when the ends are free.

The distribution of the material is found to be most effective when the bending moments at the piers and at the centre of the segment are each made equal to *one-half* of $\frac{w_1+w_2}{8}.l^2=\frac{w_1+w_2}{16}.l^2$. This will render the continuity *imperfect*, and necessitate a wedge-shaped joint.

If A is the slope of the imperfectly continuous girder, $2.A$ is the angle of the wedge, and its value is given by an equation precisely similar to (2), viz.,

$$-E.I.\tan(\pi-A)=E.I.\tan A=\frac{w_1+w_2}{24}.l^3-\frac{w_1+w_2}{16}.\frac{l^3}{2}=\frac{w_1+w_2}{96}.l^3 \qquad (7)$$

Let a_2 be the slope of the segment, when the ends are free.

$$\therefore E.I.\tan a_2=\frac{w_1+w_2}{24}.l^3,\ Ex.\ (5),\ \S\ (4).$$

Also, the slopes being always small, the angle may be substituted for their tangents.

$\therefore$ from 5, $a_1=\frac{1}{E.I}.\frac{w_2.l^3}{48}$, for a perfectly continuous girder, (9)

" 7, $A=\frac{1}{E.I}.\frac{w_1+w_2}{96}.l^3$, for an imperfectly continuous girder, (10)

" 8, $a_2=\frac{1}{E.I}.\frac{w_1+w_2}{24}.l^3$, for a girder with free ends. (11)

Let XA, XB, XC, be the tangents at X to the *continuous*, *imperfectly continuous*, and *free* girder, respectively.

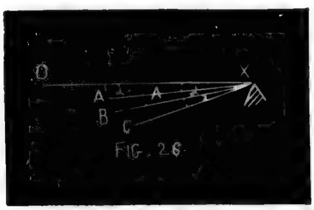


FIG. 26.

$\therefore \theta = CXB = a_2 - A$, is the angle through which the free girder must be moved to bring it into the imperfectly continuous state.

$$\therefore\ \theta=a_2-A=\frac{1}{E.I}.\frac{w_1+w_2}{24}.l^3-\frac{1}{E.I}.\frac{w_1+w_2}{96}.l^3=\frac{1}{E.I}.\frac{w_1+w_2}{32}.l^3$$

Let $\phi=CXA=a_2-a_1=\frac{1}{E.I}.\frac{w_1+w_2}{24}.l^3-\frac{1}{E.I}.\frac{w_2}{48}.l^3=\frac{1}{E.I}.\frac{2.w_1+w_2}{48}.l^3$

Hence, $\frac{\theta}{\phi}=\frac{3}{2}.\frac{w_1+w_2}{2w_1+w_2}$.

Note.–If every span is uniformly loaded with a weight w_2 per unit of length, a_1 is zero, and equation (2) gives $M_1 = \frac{w_1 + w_2}{12}.l^2$.

(17).–*Advantages and disadvantages of continuous girders.*—The advantages claimed for continuous girders are,—facility of erection, a saving in the flange material, and the removal of a portion of the weight from the centre of a span towards the piers. Circumstances, however, may modify these advantages, and even render them completely valueless. The flange stresses are governed by the position of the points of inflexion, which, under a moving load, will fluctuate through a distance dependent upon the number of intermediate supports, and upon the nature of the loading. In bridges in which the ratio of the dead load to the live load is small, the fluctuation is considerable, so that for for a sensible length of the main girders a passing train will subject local members to stresses which are alternately positive and negative. This necessitates a local increase of material, as each member must be designed to bear a much higher stress than if it were strained in one way only. (A common practice is to proportion each member to bear the greatest positive stress *plus* the greatest negative stress that can come upon it.)

Again, the web of a continuous girder, even under a uniformly distributed dead load, is theoretically heavier than if each span were independent, and its weight is still further increased when it has to resist the complex stresses induced by a moving load.

Hence, in such bridges, the slight saving, if there be any, cannot be said to counterbalance the extra labour of calculation and workmanship.

In girders subjected to a dead load only, and in bridges in which the ratio of the dead load to the live load is large, the saving becomes more marked, and increases with the number of intermediate supports, being theoretically a maximum when the number is infinite. This maximum economy may be approximated to in practice by making the end spans about $\frac{8}{10}$-ths the intermediate spans.

FIG 27

It is often found economical to increase the depth of the girder over the piers, as in Fig. 27, which introduces a local stiffness, and moves the points of inflexion farther from the piers. A point of

inflexion may be made to travel a short distance by raising or depressing one of the supports.

In order to ensure the full advantage of continuity the utmost care and skill are required both in design and workmanship. Allowance has to be made for the excessive expansion and contraction due to changes of temperature, and the piers and abutments must be of the strongest and best description so that there may be no settlement. Indeed, the difficulties and uncertainties to be dealt with in the construction of continuous girders are of such a serious, if not insurmountable character, that American engineers have almost entirely discarded their use except for draw-spans.

Much, in fact, is mere guess-work, and it is usual in practice to be guided by experience, which confines the points of inflexion within certain safe limits.

Under these circumstances it may prove desirable to fix the points of inflexion *absolutely*, because the calculation of web stresses then becomes easy and definite, instead of complicated and even indeterminate, as is the case when such points are movable; and also because each member is strained in one way only, and therefore requires no extra metal, as it does when subjected to stresses alternately positive and negative.

The *fixing* may be thus effected:

(*a*).–A *hinge* may be introduced at the selected point.

FIG 28.

The benefit of doing so is very obvious when circumstances require a wide centre span and two short side spans, as in Fig. 28.

(*b*).–If the web is open, *i. e.*, lattice-work, the point of inflexion in the upper flange may be fixed by cutting the flange at the selected point and lowering one of the supports so as to produce a slight opening between the severed parts. The position of the point of inflexion in the lower flange is then defined by the condition, that the algebraic sum of the horizontal components of the stresses in the diagonals intersected by a line joining the two points of inflexion is zero.

EXAMPLES.

(1).–Two angle irons, each 2-in. × 2-in. × $\frac{1}{2}$-in., were placed upon supports 12-ft. 9-ins. apart, the transverse outside distance between the bars being $9\frac{1}{2}$-ins., and were prevented from turning inwards by a thin plate as in the Fig.

The bars were tested under uniformly distributed loads, and each was found to have deflected $2\frac{5}{16}$-in. when the load over the two was 9-cwts.; find E, and the postion of the neutral axis.

(2).–Both bars in Question (1) failed together when the total load consisted of $10\frac{1}{2}$-cwts. uniformly distributed, and 3 cwts. at the centre; find the maximum stress in the metal.

(3).–An angle iron, 3-ins. × 3-ins. × $\frac{9}{16}$-in., was placed upon supports 12-ft. 9-in. apart, and deflected $1\frac{1}{2}$-in. under a load of 8-cwts. uniformly distributed, and 2-cwts. at the centre; find E and the position of the neutral axis.

(4).–The effective length and central depth of a cast-iron girder resting upon two supports were, respectively, 11-ft. 7-ins. and 10-ins.; the bottom flange was 10-ins. wide, and $1\frac{1}{4}$-in. thick; the top flange was $2\frac{1}{2}$-ins. wide and $\frac{7}{8}$-in. thick; the thickness of the web was $\frac{3}{4}$-in. The girder was tested by being loaded at points $3\frac{3}{4}$-ft. from each end, and failed when the load at each point was $17\frac{1}{2}$-tons; what were the central flange stresses at the moment of rupture?

What was the central deflection when the load at each point was $7\frac{1}{2}$-tons? (E=18,000,000-lbs., and the weight of the girder = 3,368-lbs.)

(5).–A tubular girder rests upon supports 36-ft. apart. At 6-ft. from one end the flanges are each 27-ins. wide and $2\frac{3}{4}$-ins. thick, the net area of the tension flange being 60-ins., while the web consists of two $\frac{7}{16}$-in. plates 36-ins. deep, and 18-ins. apart. Neglecting the effect of the angle irons uniting the web plates to the flange, determine the moment of resistance.

The girder has to carry a uniformly distributed dead load of 56-tons, a uniformly distributed live load of 54-tons, and a local load at the given section of 100-tons, what are the corresponding flange stresses?

(6).–A pivot rests in its *step*, and is subjected to a pressure P in the direction of its axis; find the moment of friction over the base with respect to the axis:—(1).–When the pivot is cylindrical and hollow.

(2).–When the pivot is cylindrical and solid.

(3).–When the pivot is conical.

(7).–Assuming that the normal wear at any point of a *step* is proportional to the friction, and also to the amount of surface that passes over the point in a unit of time, shew that if the vertical wear of the step is constant, its generating line is the Tractory, and find its equation. (Schiele's anti-friction pivot.) If f is the co-efficient of friction, P the load upon the pivot, and r its external radius, shew that the moment of friction $= f.P.r.$

(8).—A rigid bar is supported nearly horizontally on three slightly elastic vertical props; determine the pressures on the props.

(9).—A girder is supported at the ends and carries a number of isolated weights. If the bending moments are the same at any pair of consecutive weights, shew that the bending moment is constant for the whole interval between them.

(10).—A weight is placed upon an ordinary rectangular table which rests on the ground; calculate the pressures on the four legs, supposing the legs to be rigid as compared with the ground.

(11).—How many $\frac{7}{8}$-in. rivets are required at the given section in Question (5) to unite the angle irons to the flanges?

(12).—A yellow pine, beam 14-ins. wide and 15-ins. deep, was placed upon supports 10-ft. 9-ins. apart, and deflected $\frac{2}{5}$-in. under a load of 20-tons at the centre; find E.

What were the intensities of the normal and tangential stresses at 2-ft. from a support upon a plane inclined at 30° to the axis of the beam?

(13).—A beam is supported at the ends and bends under its own weight; shew that the upward force at the centre which will exactly neutralize the bending action is equal to $\frac{5}{8}$ or $\frac{1}{2}$ of the weight of the beam, according as the ends are *free* or *fixed*.

Find the neutralising forces at the quarter spans.

(14).—A beam 8-ins. wide and weighing 50-lbs. per cubic ft. rests upon supports 30-ft. apart; find its depth so that it may deflect $\frac{3}{4}$-in. under its own weight. (E=1,200,000-lbs.)

(15).—A rectangular girder of given length and breadth rests upon two supports and carries a weight P at the centre; find its depth so that the elongation of the lowest fibres may be $\frac{1}{1400}$th of the original length.

(16).—A yellow pine beam, 1[illegible]-ins. wide, 15-ins. deep, and weighing 32-lbs. per cubic ft., was placed upon supports 10-ft. 6-ins. apart. Under uniformly distributed loads of 59,734-lbs., and of 127,606-lbs., the central deflections were, respectively, .18-ins. and .29-ins.; find the mean value of E.

Also determine the additional weight at the centre which will increase the deflection by $\frac{1}{10}$th of an inch.

(17).—A pitch pine beam, 14-ins. wide, 15-ins. deep, and weighing 45-lbs. per cubic ft., is placed upon supports 10-ft. 9-ins. apart, and carries a load of 20-tons at the centre; find the deflection and curvature, E being 1,270,000-lbs.

What amount of uniformly distributed load will produce the same deflection?

(18).—A sample cast-iron girder for the Waterloo Corn Warehouses, Liverpool, 20-ft. $7\frac{1}{8}$-ins. in length and 21-ins. in depth (total) at the centre, was placed upon supports 18-ft. $1\frac{1}{8}$-ins. apart, and tested under a uniformly distributed load. The top flange was 5-ins. × $1\frac{1}{4}$-in., the bottom flange was 18-ins. × 2-ins., and the web was $1\frac{1}{4}$-in. thick. The girder deflected .15-ins., .2-ins., .25-ins., and .28-ins., under loads (including weight

of girder) of 63,763-lbs., 88,571-lbs, 107,468 lbs., and 119,746-lbs., respectively, and broke during a sharp frost under a load of 390,282-lbs.; find the mean co-efficient of elasticity, and the central flange stresses at the moment of rupture.

Is it *probable* that the girder failed by the crushing of the top flange or by the tearing of the bottom flange?

(19).–A steel rectangular girder, 2 ins. wide, 4-ins. deep, is placed upon supports 20-ft. apart; if E is 35,000,000-lbs., find the weight which, if placed at the centre, will cause the beam to deflect 1-inch.

(20).–A girder of uniform section may have both ends fixed, one end fixed and one free, or both ends free; compare its strength under these different conditions, when it has to carry, (1).–a uniformly distributed load, (2).–a single weight at the centre.

Shew that the corresponding *central* deflections are as 1:2:5 for the uniformly distributed load and as 4:7:16 for the single weight.

Find the maximum deflection in each case.

(21).–Draw the *Shearing Force* and *Bending Moment* diagrams in each case of the preceding example, and compare the *theoretical* amounts of metal required in the several webs and flanges.

(22) –A timber girder of uniform section carrying a single weight at the centre and with both ends *fixed*, is *theoretically* twice, *practically* $1\frac{1}{2}$-times as strong as if both ends were *free:* also, the stresses at the ends of the fixed girder are *theoretically* the same as at the centre, but are *practically* small, and the fracture invariably takes place at the centre; explain wherein the theory is defective.

(23).–A girder of *uniform strength*, of length l, breadth b, and depth d, rests upon two supports, and carries a uniformly distributed load of w-lbs. per unit of length which produces an inch-stress of f-lbs. at every point of the material. Shew that the central deflection

$$\text{is } \frac{\pi-2}{2}\cdot\frac{f^{\frac{3}{2}}}{E}\cdot\left(\frac{b}{3.w}\right)^{\frac{1}{2}}\cdot l$$

when b is constant and d variable. Find the deflection when d is constant and b variable.

Would it be in accordance with the true principles of construction to make the sectional area, *i.e.*, *b.d.*, equal to a constant quantity? Why?

(24).–A semi-girder of *uniform strength*, of length l, breadth b, and depth d, carries a weight W at the free end which produces an inch-stress of f-lbs. at every point of the material; prove that the maximum deflection is $\frac{4}{3}\cdot\frac{(f.l)^{\frac{3}{2}}}{E}\cdot\left(\frac{b}{6.W}\right)^{\frac{1}{2}}$ when b is constant and d variable, and that it is twice as great as it would be if the section were uniform throughout and equal to that at the support.

What would be the maximum deflection if the semi-girder were subjected to a uniformly distributed load of w-lbs. per unit of length?

(25).–A uniform load $P\,Q$ moves along a horizontal beam resting

upon supports at its ends A and B; prove that the bending moment at a given point O is a maximum when PQ occupies such a position that

$$\frac{OP}{OQ}=\frac{OA}{OB}.$$

Draw curves of maximum shearing force and bending moment for all points of the girder.

(26).–The flange of a girder consists of a pair of angle-irons and of a plate which extends over the middle portion of the girder for a certain required distance; shew that the greatest economy of material is secured, when the length of the plate is $\frac{2}{3}$-rds of the span, and the sectional areas of the plate and angle-irons are as 4 to 5. (The girder being uniformly loaded.)

(27).–The flange of a uniformly loaded girder is to consist of two plates, each of which extends over the middle portion of the girder for a certain required distance, and of a pair of angle-irons; shew that the greatest economy of material is realised when the lengths of the plates and angle-irons are in the ratio of 12:18:23, and when the areas of the plates are in the ratio of 4:5.

What should be the relative length of the plates, if they are of equal sectional area?

(28).–An elastic beam rests upon supports at its ends, and a weight placed at a point A produces a certain deflection (d) at a point B; shew that if the weight is transferred to B the same deflection (d) is produced at A.

(29).–A uniform elastic beam is supported in a horizontal plane by props A, B, C, ...; if w is the weight of the beam per unit of length, the curve assumed by the neutral axis AB is

$$y=\frac{w.x^4}{24.E}+\frac{a.x^3}{6}+\frac{b.x^2}{2}+c.x+d,$$

a, b, c, d, being constants.

Shew how to find the values of a, b, c, d.

(30).–A single weight travels over the span AB of a girder of two equal spans AB, BC, continuous over a centre pier B, shew that the reaction at C is a maximum when the distance of the weight from A is $\frac{AB}{\sqrt{3}}$, and find the corresponding bending moment at the central pier.

(31).–AB, BC are two equal spans of a girder continuous over a centre pier B. Two locomotives are crossing the girder, the one on AB, and the other on BC. If W is the weight upon an axle, and if at any given time p is the distance from A of an axle of the locomotive on AB, and q the distance from C of an axle of the locomotive on BC, shew that the bending moment at the central pier is given by

$$M=\frac{1}{4.l^2}.\left\{\Sigma W.p(l^2-p^2)+\Sigma W.q.(l^2-q^2)\right\}$$

(32).–Trace the curves which will shew the maximum shearing force and bending moments at the different sections of the span AB in the two preceding questions, as the live load travels across the girder.

(33).–A girder, continuous over several supports, is arbitrarily loaded; if l_r, l_{r+1}, are two consecutive spans, A_r, A_{r+1}, the areas of the corresponding bending moment curves, and z_r, z_{r+1}, the distances of the centres of gravity of the curves from the corresponding outer supports, shew that the Theorem of Three Moments may be expressed in the form,

$$M_{r-1}.l_r + 2.M_r.(l_r + l_{r+1}) + M_{r+1}.l_{r+1} = -6.A_r.\frac{z_r}{l_r} - 6.A_{r+1}.\frac{z_{r+1}}{l_{r+1}}$$

(34).–A uniform beam is supported by *four* equidistant props, of which *two* are terminal; shew that the two points of inflexion in the middle segment are in the same horizontal plane as the props.

(35).–Each of the main girders of the Torksey Bridge is continuous, and consists of two equal spans, each 130-ft. long. The girders are double-webbed, the thickness of each web plate is $\frac{1}{4}$-in. at the centre and $\frac{3}{8}$-in. at the abutments and centre pier; the total depth of the girders is 10-ft., and the depth from centre to centre of the flanges is 9-ft. $4\frac{3}{8}$-ins.

Find the reactions at the supports, and also the points of inflexion, when 200-tons of live load cover *one* span, the total dead load upon each span being 180-tons uniformly distributed.

The top flange is cellular; its *gross* sectional area at the centre of each span is 51-sq. ins., and the corresponding *net* sectional area of the bottom flange is 55-sq. ins.; determine the flange stresses and the position of the neutral axis. (According to Dr. Pole $I = 372{,}500$.)

(36).–Two tracks, 6-ft. apart, cross the Torksey Bridge, and are supported by single-webbed plate cross-girders 25-ft. long and 14-ins. deep. If the whole of the weight upon a pair of drivers, viz. 10-tons, be directly transmitted to one of these cross-girders, draw the corresponding shearing force and bending moment diagrams, (1).–if the ends of the cross-girder are *fixed* to the bottom flanges of the main girders, (2).–if they merely rest on the said flanges.

Find the *maximum* deflection of the cross-girder and the *work done* in bending it, in each case.

(38).–A girder consists of two spans AB, BC, each of length l, and is continuous over a centre pier B. A uniform load of length $2.a$ ($<l$) and of intensity w travels over AB; find the reactions at the supports for any given position of the load, and shew that the bending moment at the centre pier is a *maximum* and equal to $\frac{a.w.l}{3\sqrt{3}}\left(1 - \frac{a^2}{l^2}\right)^{\frac{3}{2}}$, when the centre of the load is at a distance $\left(\frac{l^2 - a^2}{3}\right)^{\frac{1}{2}}$ from A.

(39).–A swing bridge consists of the tail-end AB, and of a span BC, of length l-ft., the pivot being at B. The ballast box, of weight W, extends over a length AD ($= 2.c$-ft.), and the weight of the bridge from D to B is w tons per lineal ft. If $DB = x$, if p is the cost per ton of the bridge, and if q is the cost per ton of the ballast, shew that the total cost is a *minimum* when $x = \left(q.\frac{l^2 - c^2}{2.p - q}\right)^{\frac{1}{2}}$, and that the cor-

responding weight of the ballast is $w.x.\left(\frac{p}{q} - 1\right) + w.c - W.$

(40).–Compare, *graphically*, the shearing forces and bending moments along the span BC of the bridge in the preceding question when the bridge is closed, with their values when the bridge is open.

What provision should be made to meet the change in the *kind* of stress ?

(41).–A continuous girder rests upon three supports, and consists of two unequal spans AB $(= l_1)$, BC $(= l_2)$. A uniform load of intensity w travels over AB, and at a given instant covers a length AD $(= r)$ of the span. If R_1, R_3, are the reactions at A and C respectively, shew that $R_1.l_1^2 + R_3.l_2^2 = w.r.\left(l_1^2 - \frac{3}{4}.r.l_1 + \frac{1}{8}.\frac{r^3}{l_1}\right)$

Draw a diagram shewing the shearing force in front of the moving load as it crosses the girder.

(42).–If the live load in the preceding question may cover both spans, shew that the shearing force at any point D is a maximum when AD and BC are loaded, and BD unloaded.

Illustrate this force, *graphically*, taking into account the dead load upon the girder.

(43).–Each of the main girders of the Vina Del Mar Bridge rests upon two end supports, is continuous over five intermediate supports, is fixed at the centre support, is 3-ft. deep throughout, and is designed to carry a uniformly distributed *dead* load of 8-tons, and a live load of $\frac{1}{2}$-ton per lineal ft. The end spans are each 51-ft. 8-ins. and the intermediate spans each 50-ft. in the clear. Find the reactions at the supports.

The girders are single-webbed and double-flanged; the flanges are 12-ins. wide and equal in sectional area, the areas for the intermediate spans being 13-sq. ins. and 17-sq. ins. at the centre and piers, respectively. Find the corresponding moments of resistance and flange stresses, the web being $\frac{3}{8}$-in. thick.

(44).–A girder supported at the ends is 30-ft. in the clear and carries two stationary loads, viz., 7-tons concentrated at 6-ft., and 12-tons at 18-ft. from the left support; find the position and amount of the maximum deflection, and also the *work of flexure*.

The girder is built up of plates and angle-irons, and is 24-ins. deep; if the moment of resistance due to the web is neglected, and if the intensity of the longitudinal stress is not to exceed 5-tons per sq. in., what should be the flange sectional area corresponding to the maximum bending moment.

(45).–Determine the *work of flexure* and the necessary flange sectional area at the centre if the girder in the preceding question is subjected to a uniformly distributed load of 40-tons, instead of the isolated loads.

(46).–(*a*).–The bridge over the Garonne at Langon carries a double track, is about 695-ft. in length, and consists of three spans AB, BC, CD. The two main girders are continuous, and rest upon the abut-

ments at A and D, and upon the pier at B, but are *fixed* to the pier at C, so that they are free to slide over the supports at A, B and D, under changes of temperature. The *effective* length of each of the spans AB, CD, is 208-ft. 6-ins., and of the centre span BC, 243-ft. The permanent load upon a main girder is 1277-lbs. per lineal ft., and the *proof* load is 2688-lbs. per lineal ft. Find the reactions at the supports, (1).–when the proof load covers the span AB, (2).–when the proof load covers the span BC, (3).–when the proof load covers the spans AB and BC, (4).–when the proof load covers the spans BC and CD, (5).–when the proof load covers the whole girder.

Draw shearing force and bending moment diagrams for each case.

(b).–At the piers the web is $\frac{1}{2}$-in. thick and 18-ft. in depth, and each flange is made up of *four* plates $\frac{1}{2}$-in. thick and 3-ft. wide; determine the flange stresses for cases (1) and (2).

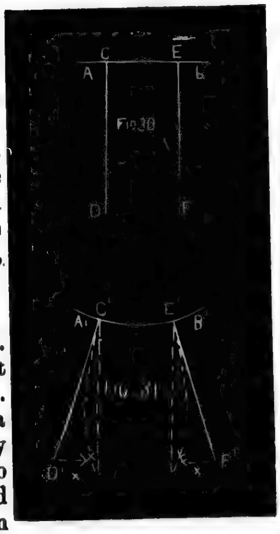
(c).–The angle-irons connecting the flanges with the web at the pier. are riveted to the former with $1\frac{1}{8}$-in. rivets, and to the latter with 1-inch rivets; how many of each kind are required in one line lineal fts on both sides of the pier at B?

(d).–The *effective* height of the pier at B is 41-ft., its mean thickness is 14-ft. 9-ins., its width is 42-ft. 9-ins., and it weighs 125-lbs. per cubic ft. If there is no surcharge on the bridge, and if the coefficient of friction between the *sliding* surfaces at the top of the pier is taken at .15, shew that the overturning moment due to the dilatation of the girders is about $\frac{1}{10}$th of the moment of stability of the pier.

(e).–Find the *points of inflexion* and also the *maximum deflections* in the *four* cases.

What practical advantage is derived from the calculation of the deflection?

(47).–The two columns $C\,D$, $E\,F$, l-ft. in length and $2.a$-ft. apart, support a cross girder over which passes a railway track of gauge $A\,B$ ($2.c$-*ft.*). The heaviest load transmitted to the columns is W-tons on each line of rail, and it is desired that under this load the direction of the resultant thrust upon a column shall coincide with its axis. To ensure this the frame is constructed as follows:—A camber is given to the girder, Fig. 2, equal in amount to the deflection of the girder $A\,B$, Fig. 1, under the load W, *i. e.*, $\dfrac{W.a^2.\overline{c\text{-}a}}{2.E.I}$



I being the moment of inertia of the *girder*. The columns are rigidly fixed to the girder at right angles to the tangents at C' and E'. Finally, the ends D' and F' are pulled by a horizontal force X until they are vertically beneath the points C' and E', respectively, so that the columns assume the dotted curved forms as shewn in the Fig. Evidently, when the loads W are transmitted through the columns, the required condition is fulfilled.

If M_o, M_l, are the moments of resistance at the foot and head, respectively, of either column, and if I is the moment of inertia of a column, shew that, $X.\ l=\dfrac{6.I.W.a.\overline{c-a}}{I.l+4.I_1.a}=3.M_o=\dfrac{3}{2}\cdot M_l$, and that the slope at A *or* B of the girder *or* column is $\dfrac{W.a.l.\overline{c-a}}{E.(I.l.+4.I_1.a)}$

(48).–Discuss the frame in the preceding Question, and obtain corresponding results, when the lines of rail fall *within CE.*

(49).–The section of a given beam is in the form of an isosceles-triangle, with its base horizontal; shew that the moment of resistance of the *strongest* trapezoidal beam that can be cut from it is very nearly equal to $\dfrac{11}{240}\cdot f.\ b.\ d^2$, b being the width of the base, d the depth of the triangle, and f the unit stress in the extreme layer.

(50).–The floor-beams of a single-track bridge are 14-ft. in the clear, and carry rolled iron joists under the rails. Determine the most economical spacing for the floor-beams on the assumptions, (1).–that the weight per lineal yd. of the rolled beam is to be proportional to the square root of the maximum bending moment upon it, (2).–that the floor-beams are to be 18-ins. deep with a $\frac{3}{8}$-in. web, and a flange sectional area proportional to the maximum bending moment, and (3).–that the live load is equivalent to a distributed load of $1\frac{1}{2}$-tons per lineal ft.

(51).–A continuous girder consists of two spans, each 50-ft. in length; the effective depth of the girder is 8-ft. If one of the end bearings settles to the extent of 1-in., find the maximum increase in the flange and shearing stress caused thereby, and shew by a diagram the change in the distribution of the stresses throughout the girder. (Assume the section of the girder to be uniform, and take $E=25{,}000{,}000$ lbs.)

(52).–A bridge, a-ft. in the clear, is formed of two cantilevers which meet in the centre of the span, and are connected by a bolt capable of transmitting a vertical pressure from the one to the other. A weight W is placed at a distance b from one of the abutments; find the pressure transmitted from one cantilever to the other, and draw the curve of bending moments for the loaded cantilever.

(53).–Weights are placed at different points of a beam supported at the two ends. Assuming that the deflection of any point of the beam is the sum of all the deflections caused by the several weights, determine the deflection produced by a uniformly distributed load.

CHAPTER III.

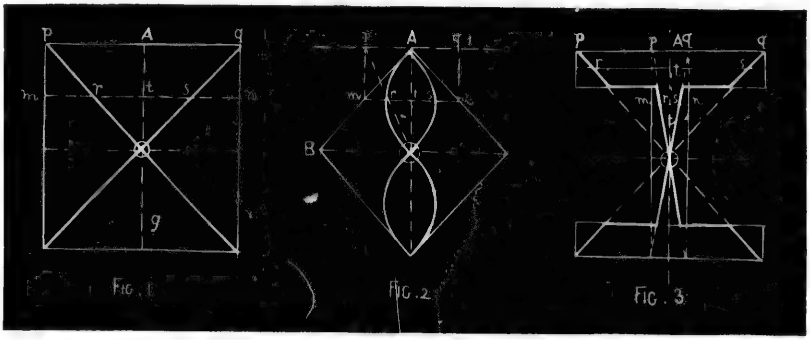
OF THE TRANSVERSE STRENGTH OF BEAMS.

(1).–*Moment of Resistance, etc.*—In the present chapter a mechanical method is described by which may be found the transverse strength of any girder having a section symmetrical with respect to a vertical axis. The method requires but a very limited knowledge of mathematics, and depends in its principle upon the condition, that the stress in any horizontal layer of a girder varies directly as the distance of the layer from the neutral axis. Of course this condition is strictly true only within the limits of elasticity, but it is often utilised in practice in finding the *ultimate* strength.

In any horizontal layer take a line, bisected by the vertical axis, proportional or equal to the stress in that layer. The locus of the extremities of all such lines encloses an area A, which evidently represents the sectional stress in magnitude and distribution. The two parts into which the area is divided by the neutral axis are equal, for, within the limits of elasticity it may be assumed that the resistance to compression is equal to the resistance to extension. Also, the resultant compressive and tensile stresses must necessarily act at the centres of gravity of the two parts.

Hence, if d is the distance between the two centres of gravity, *i. e., the effective depth of the section*, and if f is the actual stress in either of the extreme layers, then $f.\frac{A}{2}.d$ is *the moment of resistance of the section*, and A is the *effective*, or *safe resistance* area.

(2).–*Example of Regular Sections.*—Figs. 1, 2, and 3 are symmetrical with respect to a horizontal as well as a vertical axis. Consider any layer mn, and let pq be its projection upon the horizontal through A.



If f be the unit stress at A, the stress in the layer mn is $f.mn\frac{Ot}{OA}$ $=f.pq.\frac{Ot}{OA}=f.rs$, so that rs is the equivalent length at t, and the locus of all such points as r, s, encloses the *effective* area. This area, in Figs. 1 and 3, is bounded by straight lines, and in Fig. 2 by four equal parabolas, all being shewn by thick lines.

Let h be the depth and b the extreme breadth of sections 1 and 2.

The effective area of section $1=A=\frac{b.h}{2}$, and of section $2=A=\frac{b.h}{12}$.

The effective depth of section $1=d=\frac{2}{3}.h$, and of section $2=d=\frac{h}{2}$.

Hence, the moment of resistance of section $1=f.\frac{b.h^2}{6}$ and of section 2 $=f\cdot\frac{b\,h^2}{48}$.

In the same manner the moment of resistance of section 3 may be found, but a simpler method will be presently indicated.

Cor. To trace the parabolas in Fig. 2:—

Let $ot=x$, $tr=y$.

$$\therefore \frac{y}{x}=\frac{rt}{ot}=\frac{p\,A}{OA}=\frac{mt}{OA}=\frac{OB}{OA}\cdot\frac{At}{OA}=\frac{b}{h}\cdot\frac{\frac{h}{2}-x}{\frac{h}{2}}.$$

$$\therefore\ y=\frac{2.b}{h^2}\cdot\left(\frac{h}{2}\cdot x-x^2\right) \text{ is the equation required.}$$

(3).—*Examples of Irregular Sections.*—Figs. 4, 5, and 6 are symmetrical with respect to a vertical axis only, but the method is precisely the same.

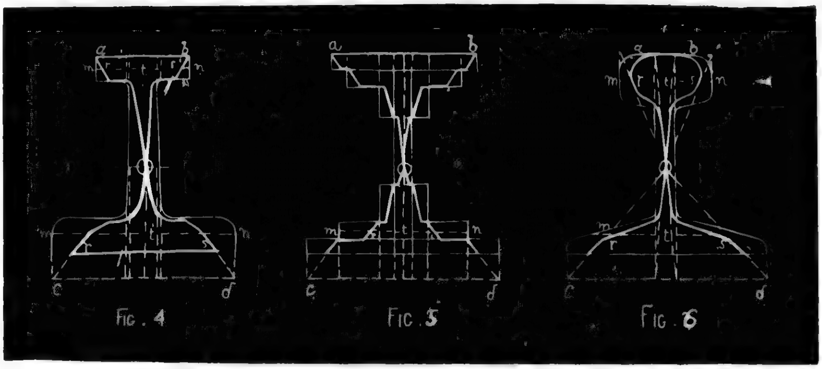


Fig. 4 Fig. 5 Fig. 6

Let O be the centre of gravity of the section, and let ab be the most extreme layer; let cd be a horizontal line at the same distance from O as ab, but on the opposite side.

Consider any layer mn, and project it upon ab or cd, according as it is situated above or below O. Join the feet of the projections with O, intersecting mn in r and s. The equivalent length of mn is rs as before, and the locus of the points r, s, (shewn by the thick lines), bounds the effective area.

Corollary.–Let f_t, f_p, be the safe tensile and compressive unit stresses, respectively.

Let d_t, d_p, be the distances from O of the extreme fibres in tension and compression respectively.

The beam tends to fail by tension or compression according as

$$\frac{f_t}{f_p} < \text{or} > \frac{d_t}{d_p}$$

(4).–*Example of Hollow Circular Section.*—The effective area of a hollow circular section may be found by treating the section as the difference of two solid circles.

The stress in any layer MN $= f.MN.\dfrac{OT}{OA} = f.2.RT$, which fixes the point R for the outer circle.

Let $RT=y$, and $TO=x$; let h be the diar. of the outer circle, and h that of the inner.

$$\therefore \frac{y}{x}=\frac{RT}{OT}=\frac{PA}{OA}=\frac{MT}{OA}=\frac{\sqrt{\frac{h^2}{4}-x^2}}{\frac{h}{2}}$$

$\therefore y=\frac{2.x}{h}\cdot\sqrt{\frac{h^2}{4}-x^2}$, is the equation to the locus of R, *i.e.*, ARO.

In the same manner may be obtained the locus $A'\,R'\,O$ for the inner circle, and its equation is $y=\frac{2.x}{h'}\cdot\sqrt{\frac{h'^2}{4}-x^2}$

(5).–*General Method* for a section symmetrical with respect to a vertical axis.

(a).–Determine its area, by cutting an accurate template of the section out of cardboard or thin metal, and carefully balancing it against a rectangle of the same material; the area of the rectangle is evidently the same as that of the section.

The area may also be obtained with a planimeter.

(b).–Determine the centre of gravity, *i.e.*, the neutral point, of the template either by balancing it upon a needle point, or by the method of suspension, and transfer it to a drawing of the section.

(c).–Trace the bounding lines of the effective area as follows:—

Divide up the section by a number of equidistant lines, as in Fig. 8, and project these lines, according as they are below or above the neutral point O, upon the most extreme layer AB and upon the horizontal line CD at the same distance from O as AB.

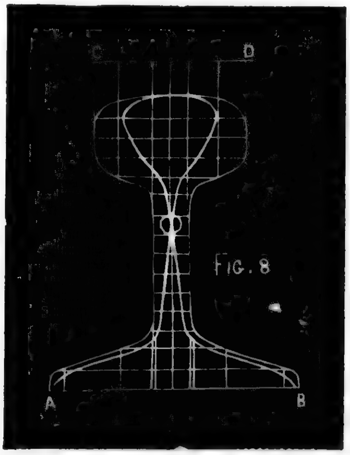


FIG. 8

The points of intersection of the lines with the radial lines from O to the extremities of the projections, define the effective area.

(d).–Cut out accurate templates of the parts of the effective area above and below O. The templates must exactly balance, and the area and centre of gravity of each may be found as in a and b.

(*e*).–Transfer the centres of gravity to the drawing; the distance between them is the effective depth.

(6).–*Practical Conclusions.*–Experiments indicate that the actual deflection and strength (both elastic and ultimate) of iron and steel girders under transverse loads, exceed the results of theory by an amount which varies from *zero* per cent. for thin webbed steel plate girders, to 60 and 70 per cent., respectively, for solid rectangular bars of wrought-iron and steel. The excess for other sectional forms is intermediate between these extreme limits, and for any given section may be assumed to be the product of $\frac{60}{100}$ or $\frac{70}{100}$ by the ratio of the area of the section to the area of the rectangle formed by the width of the lower face and the depth of the girder.

Denote the last ratio by r.

Let a be the *effective flange* area.

Let d be the *effective* depth.

Let f_1 be the *theoretic* unit-stress in a girder under a load of W-tons at the centre of x-inch bearings.

Let f_2 be the *actual* direct strength of the material of the girder.

$$\therefore f_1 . a.d = \frac{W.x}{4}$$

$$\text{and, } f_1 = f_2 . \left(1 + \frac{p.r}{100}\right)$$

p being 60 or 70 according as the girder is of wrought-iron or steel.

$$\text{Hence, } f_2 . \left(1 + \frac{p.r}{100}\right) . a.d = \frac{W.x}{4}$$

an equation which gives the strength of a girder within a comparatively small percentage of its actual value, provided the rules in the preceding article are carefully applied, and that failure does not occur from local weakness.

A girder, loaded transversally, may fail from lateral flexure long before its ultimate strength is reached, either because the upper flange is narrow and unsupported by a thick web, or because the web is thin and elastic although the width of the flange may be ample. No general theory makes allowance for failure from such causes, and they must be guarded against by *practical* considerations. In practice, the *elastic* strength of a beam free from local weakness averages from 50 to 55 per cent. of its *ultimate* strength.

EXAMPLES.

(1).–Shew that the ratio of the strength of a square beam with its side vertical to the strength of the same beam with a diagonal vertical is $\sqrt{2}$.

Explain why such a result might be anticipated.

(2).–Compare the strength of a T-section when the flange is uppermost with the strength of the same section when the web is uppermost.

(3).–Determine the resistance area of a regular cruciform section.

(4).–Shew that the effective area and effective depth of a circular section of diameter h are $\frac{h^2}{3}$ and $\frac{3}{16}.\pi.h$, respectively.

(5).–Find the effective area and effective depth of a hollow circular section.

(6).–The flanges of an I-section are each $3\frac{1}{2}$-ins. wide and 1-in. thick, the web is 4-ins. deep and 1-in. thick, the safe tensile and compressive working stress is 5-tons per sq. in.; determine the moment of resistance.

(7).–The top flange of a cast iron girder is 5-ins. wide and 2-ins. thick, the bottom flange is 18-ins. wide and 2-ins. thick, and the web is 23-ins. deep and 2-ins. thick; determine the greatest tensile stress in the section corresponding to a compressive stress in the extreme layer of $1\frac{1}{2}$-tons per sq. in.

(8).–Each of the flanges of a wrought-iron girder con- 's of *five* 8-in. by $\frac{1}{2}$-in. plates, riveted to a 24-in. by $\frac{1}{2}$-in. web, by two by 3-ins. by $\frac{1}{2}$-in. angles; the ultimate tenacity and the elastic streng... the iron are 21-tons and $10\frac{1}{2}$-tons per sq. in., respectively; find the weight which the girder will support without appreciable set at the centre of 20-ft. bearings, the moment of resistance being 521-inch-tons, or the mean of the moments of resistance in tension and compression.

(9).–*Eight* $5\frac{1}{2}$-ins. by $\frac{1}{2}$-in. plates and two $2\frac{1}{2}$-ins. by $2\frac{1}{2}$ ins. by $\frac{1}{2}$-in. angles were substituted for the bottom flange in the preceding question, when the girder failed by distortion under a load of 102-tons at the centre of 20 ft. bearings. The compressive and tensile moments of resistance were 630 and 450-inch-tons, respectively; what were the corresponding inch-stresses?

Explain the cause of the failure.

(10).–The top flange of a cast-iron girder was 3-ins. wide by $\frac{3}{4}$-in. thick, the bottom flange was 9-ins. wide by $\frac{9}{16}$-ins. thick, the web was 11 ins. deep by $\frac{3}{4}$-in. thick; the greatest working stress in compression was 4-tons per sq. in., what was the corresponding greatest tensile stress?

(11).–The girder in the preceding question was placed upon supports 11-ft. 6-ins. apart, and failed under a load of 42-tons at the centre; determine the equivalent direct tensile strength of the iron.

(12).–A channel beam, 3-ins. by $1\frac{3}{4}$-in. by $\frac{3}{8}$-in., is made of wrought-iron having an elastic limit of 10-tons per sq. in., and an ultimate tena-

city of 20 tons per sq. in.; determine the weight which the beam will support without appreciable set at the centre of 36-inch bearings, the web being uppermost.

(13).–A steel channel beam consists of a $6\frac{1}{4}$-in. by $\frac{3}{8}$-in. web and two $2\frac{3}{4}$-ins. by $2\frac{3}{4}$-ins. by $\frac{5}{16}$ in. angles; the beam is to bear, without appreciable set, a weight of 10,500 lbs. at the centre of 36-inch bearings, and it is specified that the ultimate tenacity of the metal is to be 29 tons per sq. in.; what is its elastic strength?

(14).–The angles in the preceding question are placed back to back at the centre of the plate so as to form a T-section; determine the weight which the beam will bear at the centre of 36-inch bearings, without appreciable set, (1).–When the plate is uppermost.
(2).–When the angles are "

(15).–Determine the effective area and effective depth of a Sandberg standard 60-lb. flanged steel rail, $4\frac{1}{2}$-ins. deep by 4-ins. wide.

The ultimate tenacity of the steel is to be 40-tons per sq. in. and the elastic limit 50 per cent.; find the greatest transverse strength of the rail at 42-inch bearings.

If the minimum tenacity of the steel is 32-tons per sq. in., find the corresponding elastic limit and transverse strength.

(16).–The sectional area, effective area, and effective depth of a steel double-headed rail are 5·84-sq. ins., 3·31-sq. ins., and 2·94-ins., respectively; the steel has an ultimate tensile strength of 43-tons per sq. in. and an elastic limit of 54 per cent.; what weight will the rail bear at the centre of 60-inch bearings, without permanent set?

(17).–The sectional area, effective area, and effective depth of a steel double-headed rail are 8·05-sq. ins., 4·8-sq. ins., and 4·05-sq. ins., respectively, and the rail carries an ultimate load of 35-tons at the centre of 60-inch bearings; determine the equivalent direct tensile strength of the steel.

(18).–The effective tension and compression areas of an iron rail are 2·6-sq. ins. and 2·3-sq. ins., respectively; the effective depth is 3·6-ins.; the ultimate tenacity of the iron is 25-tons per sq. in.; find the corresponding compressive unit stress, and also the ultimate transverse strength of the rail at 48-inch bearings.

(Ratio of sectional area to enclosing rectangle = ·32.)

(19).–The flanged rails of the Canadian Pacific Ry. have a sectional area of 5·7-sq. ins., an effective area of 4-sq. ins., and an effective depth of 3-ins.; what weight will one of these rails support, without permanent set, at the centre of 54-inch bearings, (1).–if it is made of steel having an ultimate tenacity of 35-tons per sq. in., and an elastic limit of 45 per cent, (2).–if it is made of wrought iron having an ultimate tenacity of 25-tons per sq. in. and an elastic limit of 50 per cent.

(20).–The sectional area, effective area, and effective depth of a flanged steel rail are 7·1-sq. ins., 4·5-sq. ins. and 3·6-ins., respectively; the rail just bears a weight of 17-tons at the centre of 60-inch bearings

without permanent set; find the ultimate tensile strength of the steel, its elastic limit being 54 per cent.

(21).–The maximum and minimum tensile strength of a steel flanged rail are 37-tons and 33-tons per sq. in., respectively, and the corresponding elastic limits are 53 per cent. and 60 per cent.; determine the maximum and minimum transverse strength of the rail at 42-inch bearings.

(Moment of resistance of section $= 8\frac{3}{4}$-inch-tons, ratio of sectional area to enclosing rectangle $= \cdot 31$.)

(22).–If the sectional area of the table of a Barlow rail is equal to the product of $\cdot 273$ by the joint area of the quadrantal wings, shew that the sectional area of the rail is $4.r.t$ nearly, and also that the moment of resistance is $\frac{8}{7}.f.t.r^2$, nearly.

($t =$ thickness of quadrantal wings, and $r =$ their radius measured to the middle of the thickness.)

CHAPTER IV.

PILLARS.

(1).–*Classification.*—The manner in which a material fails under pressure depends not merely upon its *nature* but also upon its *dimensions* and *form*. A short pillar, *e.g.*, a cubical block, will bear a weight that will almost crush it into powder, while a thin plank or a metal coin subjected to enormous compression will be only condensed thereby. In designing struts or posts for bridges and other structures, it must be borne in mind that such members have to resist *buckling* and *bending* in addition to a direct pressure, and that the tendency to buckle or bend increases with the ratio of the length of a pillar to its least transverse dimension.

Hodgkinson, guided by the results of his experiments, divided *all pillars with truly flat and firmly bedded ends* into *three* classes, viz. :—

(A).–*Short Pillars*, of which the ratio of the length to the diameter is less than 4 or 5 ; these fail under a direct pressure.

(B).–*Medium Pillars*, of which the ratio of the length to the diameter exceeds 5, and is less than 30 if of cast-iron or timber, and less than 60 if of wrought-iron ; these fail partly by crushing and partly by flexure.

(C).–*Long Pillars*, of which the ratio of the length to the diameter exceeds 30 if of cast-iron or timber, and 60 if of wrought-iron ; these fail wholly by flexure.

(2).–*Further deductions from Hodgkinson's experiments.*—A pillar with both ends rough from the foundry so that a load can be applied only at a few isolated points, and a pillar with a rounded end so that the load can be applied only along the axis, are each *one-third* of the strength of a pillar of class B, and from *one-third* to *two-thirds* of the strength of a pillar of class C, the pillars being of the same dimensions.

The strength of a pillar with one end flat and the other round is an arithmetical mean between the strengths of two pillars of the same dimensions, the one having both ends flat and the other both ends round.

Discs at the ends of pillars only slightly increase their strength, but facilitate the formation of connections.

An enlargement of the middle section of a pillar sometimes increases its strength in a small degree, as in the case of solid cast-iron pillars with rounded ends which are made stronger by about *one-seventh* ; hollow cast-iron pillars, however, are not affected. The strength of a disc-ended pillar is increased by about *one eighth* or *one-ninth*, when the middle diameter is lengthened by 50 per cent., but for slight enlargements the increase is imperceptible.

The strength of hollow cast-iron pillars is not affected by a slight variation in the thickness of the metal, as a thin shell is much harder than a thick one. The excess above or deficiency below the average thickness should not exceed 25 per cent.

(3).–*Form.*—According to Hodgkinson, the relative strengths of long cast-iron pillars of equal weight and length may be tabulated as follows:—

(*a*).–Pillars with *flat* ends.

The strength	of a	solid	round	pillar	being	100
"	"	"	square	"	is	93
"	"	"	triangular	"	is	110

(*b*).–Pillars with round ends, *i.e.*, ends for hinging or pin connections.

The strength	of a hollow cylindrical	pillar	being	100
"	" an H-shaped	"	"	74·6
"	" a ✚-shaped	"	"	44.2

The strengths of a long solid round pillar with flat ends, and a long hollow cylindrical pillar with round ends, are approximately in the ratio of 2.3 to 1.

The *stiffest* kind of wrought-iron strut is a built tube, the section consisting of a cell or of cells, which may be circular, rectangular, triangular, or of any convenient form.

In experimenting upon hollow tubes, Hodgkinson found that, other conditions remaining constant, the *circular* was the strongest, and was followed in order of strength by the *square* in *four* compartments ⊞, the *rectangle* in *two* compartments ◫, the *rectangle* ▭ and the *square.*

The addition of a diaphragm across the middle of the rectangle *doubled* its resistance to crippling.

(4).–*Modes of failure.*—The manner in which the crushing of *short pillars* takes place depends upon the material, and the failing may be by *splitting*, *bulging*, or *buckling*.

(*a*).–*Splitting* into fragments is characteristic of such crystalline, fibrous, or granular substances as glass, timber, stone, brick and cast-iron.

A hard vitreous material, *e.g.*, glass or vitrified brick, splits into a number of prisms, (Fig. 1.)

A fibrous material, *e.g.*, timber, and granular materials, *e.g.*, cast-iron and many kinds of stone and brick, shear or slide along planes oblique to the direction of the thrust, and form one or more wedges or pyramids, (Figs. 2, 3, 4.)

Sometimes a granular or a crystalline substance will suddenly give way and be reduced to powder.

(*b*).–*Bulging*, *i.e.*, a lateral spreading out, is characteristic of fibrous materials, *e.g.*, wrought-iron, copper, lead and timber.

All substances, however, even the most crystalline, will bulge slightly before they fail, if they possess some degree of toughness.

(*c*).–*Buckling* is characteristic of fibrous materials, and the resistance of a pillar to it is always less than its resistance to direct crushing, and is independent of length.

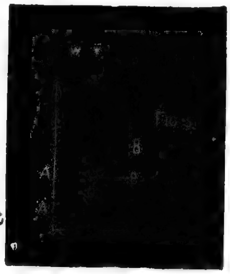
Thin malleable plates usually fail by the bending, puckering, wrinkling, or crumpling up of the fibres, and the same phenomena may be observed in the case of timber and of long bars.

Long plate tubes, when compressed longitudinally, first bend and eventually fail by the buckling of a short length on the concave side.

The ultimate resistance to buckling of a well made and well shaped tube, is about 27,000-lbs. per sq. in. section of metal, which may be increased to 33,000 or 36,000-lbs. per sq. in., by dividing the tube into two or more compartments.

A rectangular wrought iron or steel tube offers the greatest resistance to buckling, when the mass of the material is concentrated at the angles, while the sides consist of thin plates or lattice-work, sufficiently strong to prevent the bending of the angles.

Timber offers *twice* the resistance to crushing when dry than when wet, as the presence of moisture diminishes the lateral adhesion of the fibres.



(5).—*Uniform Stress.*—Let a short pillar be subjected to a pressure of W-lbs. uniformly distributed over its end, and acting in the direction of its axis.

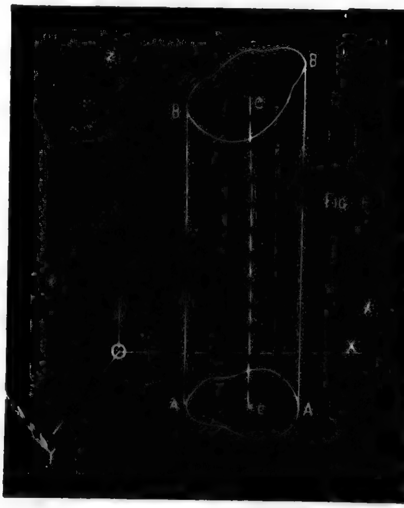
Let S be the transverse sectional area of the pillar.

Let $p = \frac{W}{S}$, be the intensity of stress per unit of area of any transverse section AB.

Let $A'B'$ be any other section of area S', inclined to the axis at an angle θ. The intensity of stress per unit of area of $A'B' = \frac{W}{S'} = \frac{W}{S}.\sin\theta$ $= p.\sin\theta$, which may be resolved into a component $p.\sin^2\theta$ normal to $A'B'$, and a component $p\sin\theta.\cos\theta$, *i.e.*, $p\frac{\sin 2\theta}{2}$, parallel to $A'B'$. The last intensity is evidently a maximum when $\theta = 45°$, so that the plane along which the resistance to shearing is least, and therefore along which the fracture of a homogeneous material would tend to take place, makes an angle of 45° with the axis.

None of the materials of construction are truly homogeneous, and in the case of cast-iron, the irregularity of the texture and the hardness of the skin cause the angle between the plane of shear and the direction of the thrust to vary from 32° to 42°.

Hodgkinson's experiments upon blocks of different materials led him to infer that the true *crushing* strength of a material is obtained when the ratio of length to diameter is at least $1\frac{1}{2}$; for a less ratio the resistance to compression is unduly increased by the friction at the surfaces between which the block is crushed.

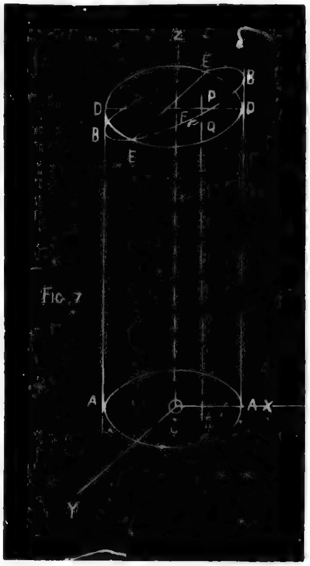


(6).—*Uniformly varying Stress.*—The load upon a pillar is rarely, if ever, uniformly distributed, but it is practically sufficient to assume that the pressure in any transverse section varies *uniformly*.

Any variable external force applied normally to a plane surface AA of area S may be graphically represented by a cylinder $AABB$, the end BB being the locus of the extremities of ordinates erected upon AA, each ordinate being proportional to the intensity of pressure at the point on which it is erected.

Let P be the total force upon AA, and let the line of its resultant intersect AA in C; C is the *centre of pressure* of AA, and the ordinate CC necessarily passes through the centre of gravity of the cylinder.

Again, the resultant *internal* stress developed in AA is P, and may of course be graphically represented by the same cylinder $AABB$.



Assume that the pressure upon AA varies uniformly; the surface BB is then a plane inclined at a certain angle to AA.

Take O, the centre of figure of AA, as the origin, and AA as the plane of $x\ y$.

Let Oy, the axis of y, be parallel to that line EE of the plane BB which is parallel to the plane AA.

Through EE draw a plane DD parallel to AA and form the cylinder $AADD$.

The two cylinders $AABB$ and $AADD$, are evidently equal in volume, and OF, the average ordinate, represents the mean pressure over AA; let it be denoted by p_o.

At any point R of the plane AA, erect the ordinate RQP, intersecting the planes DD, BB, in Q and P, respectively.

Let x, y be the co-ordinates of R.

The pressure at $R = p = PR = PQ + QR = PQ + OF = a.x + p_o$, a being a constant depending upon the variation.

Note.—The sign of x is negative for points on the left of O, and the pressure at a point corresponding to R is $p_o - a.x$.

Let x_o, y_o, be the co-ordinates of the centre of pressure C.

Let ΔS be an elementary area at any point R.

$\therefore$ $p.\Delta S$ is the pressure upon ΔS, and $\Sigma\ (p.\ \Delta S)$ is the total pressure upon the surface AA, Σ being the symbol of summation.

Hence, $x_o.\ \Sigma\ (p.\ \Delta\ S) = \Sigma\ (p.x.\Delta S)$, and $y_o.\ \Sigma\ (p.\ \Delta\ S) = \Sigma(\ p.y.\ \Delta S)$.

But $p = p_o + a.x$.

$$\therefore\ x_o.\ \Sigma\left(\overline{p_o + a.x}.\Delta S\right) = \Sigma\left(\overline{p_o.x + a.x^2}.\Delta S\right)$$

$$\text{and } y_o.\Sigma\left(\overline{p_o + a.x}.\Delta S\right) = \Sigma\left(\overline{p_o.y + a.x.y.}\ \Delta S\right)$$

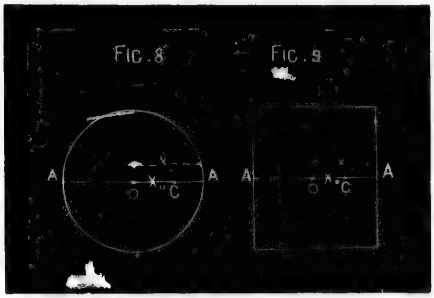
Now O is the centre of figure of AA, and therefore $\Sigma\ (x.\Delta S)$ and $\Sigma\ (y.\Delta S)$ are each zero.

Also, $\Sigma(\Delta S)=S$, $\Sigma(x^2.\Delta S)$ is the *moment of inertia* (I) of AA with respect to Oy, and $\Sigma(x.y.\ \Delta S)$ is the *product of inertia* (K) about the axis Oz.

$\therefore x_o.\ p_o.\ S = a.\ I = x_o.\ P$ (1)

And $y_o.\ p_o.S.=a.K=y_o.\ P$ (2)

Cor. 1–In any symmetrical section y_o is *zero*, and x_o is the deviation of the centre of pressure C from the centre of figure O.



Let x_1 be the distance from O of the extreme points A of the section.

The greatest stress in AA is $p_o+a.x_1=p_1$, suppose.

$$\text{But } a=\frac{x_o.\ S.p_o}{I} \text{ by (1).}$$

$$\therefore\ p_o+\frac{x_o.x_1.S.p_o}{I}=p_1$$

$$\text{or, } \frac{p_o}{p_1}=\frac{1}{1+\frac{x_o.x_1}{I.}.S.}$$

It is generally advisable, especially in masonry structures, to limit x_o by the condition that the stress shall be nowhere *negative*, *i.e.*, a tension. Now the minimum stress is $p_o-a.x_1$, so that to fulfil this condition, $p_o >$ or $=a.x_1$. But $p_1=a.x_1+p_o$, $\therefore$ $p_1 <$ or $=2.p_o$.

Hence, by (3), $\dfrac{p_o}{2.p_o} <$ or $=\dfrac{1}{1+\frac{x_o.x_1}{I}.S}$

$$\therefore \text{ and } \frac{x_o x_1.S}{I} < \text{ or}=1, \textit{ i.e., } x_o < \text{ or } =\frac{I}{x_1.S}.$$

Cor. 2.–The uniformly varying stress is equivalent to a single force P along the axis, and a couple of moment $P.CO=P\sqrt{x_o^2+y_o^2}=a.\sqrt{I^2+K^2}$

Cor. 3.–The line CO is said to be *conjugate* to Oy.

$$\text{If the angle } C\ O\ x=\theta,\ \therefore \cot.\theta=\frac{x_o}{y_o}=\frac{I}{K}.$$

(7).–*Hodgkinson's formulæ for the ultimate strength of long and medium pillars.*—When a *long* pillar is subjected to a crushing force

it first yields sideways, and eventually breaks in a manner apparently similar to the fracture of a beam under a transverse load. This similarity, however, is modified by the fact that an initial longitudinal compression is induced in the pillar by the super-imposed load.

Hodgkinson deduced, *experimentally*, that the strength of *long solid* round iron and square timber pillars, *with flat and firmly bedded ends*, is given by an expression of the form,

$$W = A.\frac{d^n}{l^m}.$$

W being the breaking weight in tons of 2240-lbs.
d " " diameter or side of the pillar in *inches*.
l " " length of the pillar in *feet*.
n and m the numerical indices.
A being a constant varying with the material and with the sectional form of the pillar.

For iron pillars, $n=3.6$ and $m=1.7$.
" timber pillars, $n=4$ and $m=2$.
" cast iron, $A=44.16$.
" wrought-iron, $A=133.75$.
" dry Dantzic oak, $A=10.95$.
" dry red deal, $A=7.81$.
" dry French oak, $A=6.9$.

The strength of *long hollow* round cast-iron pillars was found to be given by,

$$W = 44.34.\frac{d^{3.6} - d_1^{3.6}}{l^{1.7}},$$

d being the external and d_1 the internal diameter, both in inches.

Thus, the strength of a *hollow* cast-iron pillar is approximately equal to the difference between the strengths of two *solid* cast-iron pillars, whose diameters are equal to the external and internal diameters of the hollow pillar.

The strength of *medium* pillars may be obtained by the formula,

$$W' = \frac{W.f.S}{W + \frac{3}{4}f.S}.$$

W' being the breaking weight in tons of 2240-lbs.
W " " " " " as derived from the formula for *long* pillars.
f being the ultimate crushing strength in tons per sq. in.
S being the sectional area of the pillar in sq. ins.

Again, if the ends of a cast-iron pillar are rounded, the above formulæ may be still employed to determine its strength, A being 14.9 for a *solid*, and 13 for a *hollow* pillar.

(8).–*Gordon's formula for the ultimate strength of a pillar.*—The method discussed in the preceding articles being practically very inconvenient, is not generally used, and the present article will treat of Professor Gordon's formula, which has a better theoretical basis, and is easier of application.

The effect of a weight W upon a pillar of length l and sectional area S, may be divided into *two* parts:—

(a).–A *direct thrust*, which produces a uniform compression of intensity $\frac{W}{S}=p_1$

(b).–A *bending moment*, which causes the pillar to yield in the direction of its *least* dimension (h).

Let y be the greatest deviation of the pillar from the vertical.

The bending moment M at the point of maximum stress may be represented by $W.y$.

Let p_2 be the stress in the extreme layers due to this bending moment.

Now $M=\frac{p_2 I}{c}=\mu.p_2.b.h^2$,

c being the distance of the layer under consideration from the neutral axis, μ a constant depending upon the sectional form, and b the dimension perpendicular to the plane of flexure.

$$\therefore\ \mu.p_2.b.h^2=W.y, \text{ and } p_2\propto\frac{W.y}{b.h^2}.$$

But $y\propto\frac{l^2}{h}$, (§ (12), chap. II.)

$$\therefore\ p_2\propto\frac{W.l^2}{b.h.h^2}\propto\frac{W.l^2}{S.h^2}\propto p_1.\frac{l^2}{h^2} \text{ and } p_2=p_1.a.\frac{l^2}{h^2}$$

a being some constant to be determined by experiment.

Hence, the *total* stress in the most strained fibre is,

$$f=p_1+p_2=p_1.\left(1+a\frac{l^2}{h^2}\right)$$

$$\text{or, } \frac{W}{S}=p_1=\frac{f}{1+a.\frac{l^2}{h^2}}$$

which is Gordon's formula.

Cor.—If the weight upon the pillar causes the stress in any transverse section to *vary uniformly*, the *direct thrust* in the extreme layers is $\frac{W}{S}\cdot\left(1+\frac{x_0\cdot\frac{h}{2}\cdot S}{I}\right)$ instead of $\frac{W}{S}$, (Cor. 1, § (6)), x_0 being the greatest deviation of the line of resultant thrust from the axis of the pillar.

Let k be the *radius of gyration* of the cross-section, $\therefore S.k^2 = I$, and the expression for the direct thrust may be written $\frac{W}{S}\cdot\left(1+\frac{x_0.h}{2.k^2}\right)$.

Hence, Gordon's formula becomes,

$$\frac{W}{S}=p_1=\frac{f}{1+a\cdot\frac{l^2}{h^2}+\frac{x_0h}{2.k^2}}.$$

(9).—*Values of a and f.*—The following Table, giving the values of the constants a and f in Gordon's formula, has been prepared by taking an average of the best known results, and is applicable to round and square pillars *with both ends flat* :—

	f in lbs.	a
For *Cast-Iron* solid rectangular pillars	80,000	$\frac{1}{450}$
" " " round "	80,000	$\frac{1}{400}$
" " hollow rectangular "	80,000	$\frac{1}{500}$
" " " round "	80,000	$\frac{1}{600}$
For *Wrought-Iron* solid rectangular pillars............	36,000	$\frac{1}{3000}$
" " " round "	36,000	$\frac{1}{2250}$
" " thick hollow round "	36,000	$\frac{1}{5500}$
For *Mild-Steel* solid rectangular pillars...............	67,200	$\frac{1}{2000}$
" " " round "	67,200	$\frac{1}{1400}$
" " hollow round "	67,200	$\frac{1}{2500}$
For *Strong-Steel* solid rectangular "	114,000	$\frac{1}{1400}$
" " " round "	114,000	$\frac{1}{900}$
" " hollow round "	114,000	$\frac{1}{1500}$
For *Pine-Timber* solid rectangular "	6,000	$\frac{1}{250}$
" " " round "	6,000	$\frac{1}{250}$

Dry oak timber 7200 $\frac{1}{250}$

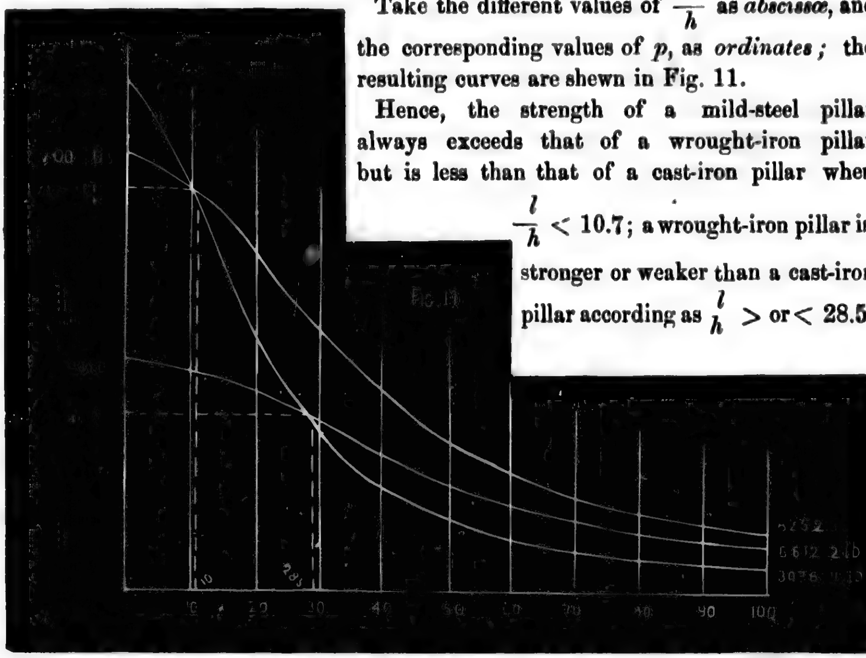
If Gordon's formula is applied to pillars with hinge pin ends *rounded at both ends*, $4.a$ takes the place of a, and if to pillars fixed at one end and rounded at the other, $\frac{9}{5}.a$ takes the place of a, (§ (1).)

(10).–*Graphical comparison of the crushing unit strength of solid round cast-iron, wrought-iron, and mild-steel pillars.*

The crushing unit stress is given by, $p = \dfrac{f}{1 + a.\frac{l^2}{h^2}}$

Take the different values of $\frac{l}{h}$ as *abscissæ*, and the corresponding values of p, as *ordinates;* the resulting curves are shewn in Fig. 11.

Hence, the strength of a mild-steel pillar always exceeds that of a wrought-iron pillar but is less than that of a cast-iron pillar when $\frac{l}{h} < 10.7$; a wrought-iron pillar is stronger or weaker than a cast-iron pillar according as $\frac{l}{h} >$ or < 28.5.



(11).–*Application of Gordon's formula to pillars of other sectional forms.*

In any section whatever, the least transverse dimension for calculation ($i.e.$, h) is to be measured in the plane of greatest flexure.

Thus, it may be taken as the *least* diameter of the rectangle circumscribing *tee* ( ), *channel* ( ), and *cruciform* ( ) sections, and as the perpendicular from the angle to the opposite side of a triangle circumscribing *angle* ( ) sections.

From a series of experiments upon *wrought-iron* pillars of these sections, f was found to be 42,500-lbs. and a, $\frac{1}{900}$.

In cast-iron struts of a cruciform section $f=80{,}000$-lbs. and $a=\frac{3}{400}$.

These results are only approximately true, and apply to pillars fixed at both ends.

Rankine's modification of G's for

(12.)–*Remarks on Gordon's formula.*—The factor a in Gordon's formula is by no means *constant*, and not only varies with the nature of the material, with the length of the pillar, with the condition of its ends, &c., but also with the sectional form of the pillar. The variation due to this latter cause may be eliminated, and the formula rendered somewhat more exact, by introducing the least *radius of gyration* instead of the least transverse dimension.

If k is the least radius of gyration,

$$\therefore k^2 = \frac{I}{mass} = \frac{m.b.h^3}{n.b.h} = \frac{m}{n}.h^2,$$

m and n being constants which depend upon the sectional form. Thus, Gordon's formula for pillars with flat ends may be written,

$$\frac{W}{S} = p_1 = \frac{f}{1 + a_1 . \frac{l^2}{k^2}}.$$ Extremely important

in which a_1 is independent of the sectional form, all variation of the latter being included in k^2.

... substitute for a, if the pillar has two pin ends $\frac{4}{5}a$, or $2a$, in case of pillar having pin ends ... and

Cor.–Gordon's formula for the strength of *solid rectangular* wrought-iron struts is,

$$\frac{W}{S} = p_1 = \frac{36{,}000}{1 + \frac{1}{3000} . \frac{l^2}{h^2}}$$

But $k^2 = \frac{h^2}{12}$,

$$\therefore \frac{W}{S} = p_1 = \frac{36{,}000}{1 + \frac{1}{36{,}000} . \frac{l^2}{k^2}},$$

Rankine gives for wrt iron $f = 36000$ $\frac{1}{a_1} = 6400$ C E fix

which may be taken as the formula for the strength of wrought-iron struts, k^2 being the square of the radius of gyration.

Ex. 1.–In a solid cylindrical pillar of diameter h, $k^2 = \frac{h^2}{16}$

$$\therefore p_1 = \frac{36{,}000}{1 + \frac{1}{2250} . \frac{l^2}{h^2}}$$

Ex. 2.–In a hollow cylinder of diameter h, $k^2=\frac{h^2}{8}$, nearly.

$$\therefore p_1=\frac{36{,}000}{1+\frac{1}{4500}\cdot\frac{l^2}{h^2}}$$

(13).–*Values of k^2 for different sections.*—

(*a*).–*Solid rectangle*:–$k^2=\frac{I}{S}=\frac{h^2}{12}$, h being the least dimension.

(*b*).–*Hollow rectangle*:–$k^2=\frac{I}{S}=\frac{1}{12}\frac{b.h^3-b'.h'^3}{b.h-b'.h'}$; b, h, being the greatest and least outside dimensions, b', h', the greatest and least inside dimensions, respectively.

Let t be the thickness of the metal, $\therefore b'=b-2.t$, and $h'=h-2.t$, and $\therefore k^2=\frac{1}{12}\cdot\frac{b.h^3-(b-2\,t).(h-2.t)^3}{b.h-(b-2\,t).(h-2\,t)}=\frac{h^2}{12}\cdot\frac{3.b+h}{b+h}$, approximately, when t is small compared with h, *i.e.*, for a *thin hollow rectangle.*

For a *square cell*, $k^2=\frac{h^2}{6}$.

(*c*).–*Solid triangle*:–$k^2=\frac{I}{S}=\frac{h^2}{18}$, h being the height.

(*d*).–*Hollow triangle*: – $k^2=\frac{I}{S}=\frac{1}{18}\cdot\frac{b.h^3-b'.h'^3}{b.h-b'.h'}$; b, h being the base and height of the outside triangle, and b', h' the base and height of the inside triangle, respectively. Also, $\frac{b}{b'}=\frac{h}{h'}$.

$$\therefore k^2=\frac{h^2}{18}\cdot\frac{b^4-b'^4}{b^2-b'^2}=\frac{h^2}{18}\cdot\left(\frac{b^2+b'^2}{b^2}\right).$$

Hence, for a *thin triangular cell*, $k^2=\frac{h^2}{9}$.

(*e*).–*Solid cylinder*:–$k^2=\frac{I}{S}=\frac{h^2}{16}$, h being the diameter.

(*f*).–*Hollow cylinder*: – $k^2=\frac{I}{S}=\frac{\pi}{16}\cdot(h^2+h'^2)$, h and h' being the external and internal diameters, respectively.

Hence, for a *thin cylindrical cell*, $k^2=\frac{h^2}{8}$, approximately.

Ex.–Gordon's formula for *hollow cylindrical cast-iron* pillars is,

$$\frac{W}{S}=P_1=\frac{f}{1+\frac{1}{500}\cdot\frac{l^2}{h^2}}=\frac{f}{1+\frac{1}{4000}\cdot\frac{l^2}{k^2}}$$

The relation $p_1 = \dfrac{f}{1 + \dfrac{1}{4000}.\dfrac{l^2}{k^2}}$ may be assumed to hold for *hollow square* struts and also for struts of a *cruciform* section.

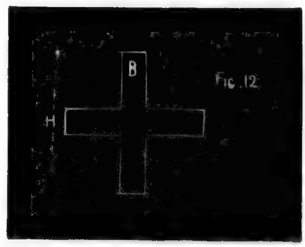
*Ex.*1.–For a hollow square having its diagonal equal to the *internal* diameter of the hollow cylinder, *i.e*, h', $k^2 = \dfrac{\left(\dfrac{h'}{\sqrt{2}}\right)}{6} = \dfrac{h'^2}{12}$

$$\text{and } p_1 = \frac{f}{1 + \dfrac{3}{1000}.\dfrac{l^2}{h'^2}}.$$

Ex. 2.–If the side of the square is equal to the *external* diameter, *i.e.*, h,

$$\therefore\ k^2 = \frac{h^2}{6}, \text{ and } p_1 = \frac{f}{1 + \dfrac{3}{2000}.\dfrac{l^2}{h^2}}$$

(*g*).–*Cruciform Section*, the arms being equal.



$$I = \frac{b.h^3}{12} + \frac{h.b^3}{12} - \frac{b^4}{12},\ S = 2.b.h - h^2$$

$$\therefore\ k^2 = \frac{1}{12}.\frac{b.h^3 + h.b^3 - b^4}{2.b.h - h^2} = \frac{h^2}{24}, \text{ nearly}$$

Hence, the formula for a cast iron pillar of cruciform section may be written,

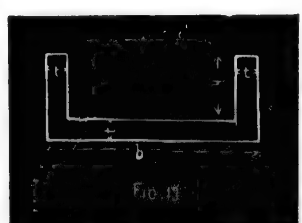
$$\frac{W}{S} = p_1 = \frac{f}{1 + \dfrac{1}{4000}.\dfrac{l^2}{k^2}} = \frac{f}{1 + \dfrac{3}{500}.\dfrac{l^2}{h^2}}.$$

(*h*).–*Angle-iron of unequal ribs* :—the greater being b and the less h.

$$k^2 = \frac{1}{12}.\frac{b^2.\,h^2}{b^2 + h^2}, \text{ approximately.}$$

Hence, if $b = h$, *i.e.*, if the ribs are equal, $k^2 = \dfrac{h^2}{24}$.

(*i*).–*Channel-iron*, the dimensions being as in Fig. 13.



$$I = \frac{b.t^3 + 2.h^3.t}{12} + \frac{2.b.h.t.^2.(h+t)^2}{4.(2.h.t + b.t)},$$

$$= h^2.\left\{\frac{2.h.t}{12} + \frac{2.b.h\,t^2}{4.(2.h.t. + b.t)}\right\}, \text{ nearly.}$$

Also, $S = b.t + 2.h.t$

$$\therefore\ k^2, = h^2.\left\{\frac{2.h.t}{12.(2.h.t + b.t)} + \frac{2\,b.h.t^2}{4.(2.h.t + b.t)^2}\right\}$$

Let the area of the two flanges$=A=2.h.t$, and let the area of the web$=B=b.t$, $\therefore\quad k^2=h^2.\left\{\frac{A}{12.(A+B)}+\frac{A.B}{4.(A+B)^2}\right\}$

(*k*).–H-*iron*, breadth of flanges being b, length of web h, and thickness of metal t.

$$I=2.\frac{b^3.t}{12}+\frac{h.t^3}{12}=2.\frac{b^3.t}{12}, \text{ nearly.}$$

$$S=2.b.t+h.t$$

$$\therefore\ k^2=\frac{b^2}{12}.\frac{2.b.t}{2b.t+h.t}=\frac{h^2}{12}.\frac{A}{A+B}.$$

A, being the area of the flanges, and B the area of the web.

(*l*).–*Circular Segment.*—Of radius r and length $r.\theta$,

$$k^2=r^2\left\{\frac{1}{2}+\frac{\sin\theta}{2.\theta}-\frac{4.\sin^2\frac{\theta}{2}}{\theta^2}\right\}$$

Hence, for a semicircle, since $\theta=\pi$,

$$k^2=r^2.\left\{\frac{1}{2}-\frac{4}{\pi^2}\right\}=\frac{r^2}{10}, \text{ nearly.}$$

(*m*).–*Barlow Rail*, proportioned as in Ex. 22, chap. III.

$$k^2=\frac{r^2}{7}, \text{ nearly.}$$

(*n*).–*Two Barlow Rails*, riveted base to base, $k^2=.393.r^2$, nearly.

(14).–*American Iron Columns.*—In 1880 Mr. G. Bouscaren read a paper before the American Society of Civil Engineers containing the results of a series of experiments made for the Cincinnati Southern *R.R.*, upon Keystone, Square, Phœnix, and American Bridge Co.'s columns.

KEYSTONE SQUARE PHOENIX AM. BRIDGE CO.

These experiments shew, as those of Hodgkinson and others have also shewn, that the strength of iron and steel columns is not only dependent on the ratio of length to diameter, and on the form of the cross-section, but also on the proportions of parts, details of design and workmanship, and on the quality of the material of which the columns are constructed.

Further, they seem to lead to the conclusions, that Gordon's formula is more correct as modified by Rankine, and that in the case of columns hinged at both ends, Rankine's formula, with a_1 assumed at double the value it has when the formula is applied to columns with flat ends, is practically correct.

The accompanying Table gives the values of the constants (a_1, and f) as deduced from Bouscaren's experiments by Prof. W. H. Burr:—

	f in lbs	a_1
For Keystone Columns with flat ends—swelled............	36,000	$\frac{1}{18,300}$
" " " " —straight (open or closed)	39,500	$\frac{1}{18,300}$
" " " " —open.... (swelled straight)	38,300	$\frac{1}{18,300}$
" " " pin ends—swelled...........	38,300	$\frac{1}{12,000}$
For Square Columns with flat ends.........	39,000	$\frac{1}{35,000}$
" " " pin ends........................	39,000	$\frac{1}{17,000}$
For Phœnix Columns with flat ends....................	42,000	$\frac{1}{50,000}$
" " round ends	42,000	$\frac{1}{12,500}$
" " pin ends........................	42,000	$\frac{1}{22,700}$
For American Bridge Co.'s Columns with flat ends..........	36,000	$\frac{1}{46,000}$
" " " round ends.........	36,000	$\frac{1}{11,500}$
" " " pin ends..........	36,000	$\frac{1}{21,500}$

In 1881 Messrs. Clarke, Reeves & Co. presented to the American Society of Civil Engineers a paper containing the results of experiments upon 20 Phœnix columns, which appeared to shew that neither Gordon's nor Rankine's formula expressed the true strength of a column of the Phœnix type. In the discussion that followed the reading of this paper, however, it was demonstrated that, within the range of the experiments, the strength of intermediate lengths and sections of Phœnix columns can be obtained either from Rankine's formula by slightly changing the constants, or from very simple new formulæ.

Mr. W. G. Bouscaren shewed that by making $a_1 = \frac{1}{100,000}$ and $f=38,000$, the calculated values of $\frac{W}{S}$ agree very nearly with the actual experimental results.

Mr. D. J. Whittemore gave the following (only applicable for lengths

varying from 5 to 45 diameters) as expressing the probable ultimate strength of these columns :—

$$W\text{-lbs} = (1200-H).30 + \frac{525,000}{H^2}$$

H being the ratio of length to diameter.

Mr. C. E. Emery stated that the ultimate strength in each case is approximately represented by the formula,

$$W\text{-lbs} = \frac{355,063 + 30,950.\ H}{H + 6.175}$$

H being the ratio of length to diameter.

Taking the different values of H as abscissæ, and of W as ordinates, this is the equation of an hyperbola. It agrees very accurately with the experimental results from 20 diameters upwards; at 15 diameters the calculated values of W are greater than those given by the experiments; for a less number of diameters the experimental results are the higher, but the variations are slight, and are provided for in the factor of safety.

The following very simple formulæ, due to Prof. W. H. Burr, give results agreeing closely with those obtained in the experiments:–

For values of $\frac{l}{k} < 30$, the ultimate strength in lbs. per sq. in.

$$= 64,700 - 4,600.\sqrt{\frac{l}{k}}$$

For values of $\frac{l}{k}$ between 30 and 140, the ultimate strength in lbs. per sq. in.

$$= 39,640 - 46.\ \frac{l}{k}$$

(k is the radius of gyration).

Note.–In designing struts, Mr. C. Shaler Smith employs the formula,

$$(\text{the working stress in lbs. per sq. in.}) \times \left(4 + \frac{H}{20}\right) = \frac{37,800}{1 + \frac{H^2}{1900}}$$

H being the ratio of length to diameter.

Thus, the factor of safety, $4 + \frac{H}{20}$, increases with H, and partially provides for the corresponding decrease in the strength to resist deflection by side blows.

(15).–*Very long thin Pillars.*–A long thin pillar of length l, and of uniform sectional area S, is bent under a weight W, and is on the point of failing from flexure.

Let h be the least transverse dimension in the plane of flexure.

Let b be the dimension in a plane perpendicular to the plane of flexure.

If the pillar is hollow, let h', b', be corresponding internal dimensions.

The most strained part of the pillar is the centre, and d may be taken as the approximate deviation from the vertical of *any* point in the central section.

$$\therefore W.d = \text{bending moment at centre} = \frac{E}{R}.I,$$

$\frac{1}{R}$ being the curvature of the pillar. But $d = \frac{l^2}{8.R}$,

and $I \propto b.h^3$, *or* $b.h^3 - b'.h'^3$ according as the pillar is solid or hollow.

$$\therefore W \propto E.\frac{b.\,h^3}{l^2}, \text{ for a long thin solid pillar.}$$

$$\text{and } W \propto E.\frac{b.h^3 - b'.h'^3}{l^2}, \quad \text{" \quad " \quad hollow \quad "}$$

Cor. 1.–$W \propto E$, so that the strength of a pillar is directly proportional to its elasticity.

Cor. 2.–If $b=h$ and $b'=h'$,

$$\therefore W \propto E.\frac{h^4}{l^2}, \text{ or } E.\frac{h^4 - h'^4}{l^2}, \text{ according as the pillar is solid or hollow.}$$

Hence, for *similar* pillars, W varies directly as the square of a linear dimension, so that the strengths of similar pillars are proportional to their sectional areas.

Cor. 3.–f, the unit stress in the outside layer, $\propto \frac{1}{R}$ (Chap. II., §(12))

$$\therefore f \propto d. \text{ Also } \frac{2f}{h}.I = M, \text{ and } \therefore M \propto f, \ i.e., \propto d.$$

But $W = \frac{M}{d}$, so that W is approximately constant, and independent of the flexure.

Cor. 4.–Let F and F', be the resultant compressive and tensile stresses at the centre.

$$\therefore F - F' = W,$$

an equation which will indicate whether a pillar will tend to fail by crushing or tearing.

Note.–The agreement of the preceding results with practice depends upon, (1).–the accuracy of the equation of moments, (2).–the accuracy of the equation of deflection, (3).–the accuracy of the law of elasticity.

(16).–*Weyrauch's theory of the resistance to buckling.*

In order to make allowance for buckling, Weyrauch proposes the two following methods :–

Method I.–Let F_1 be the necessary sectional area, and b_1 the admissible unit stress for a strut subjected to loads varying from a maximum compression B_1 to a minimum compression B_2.

Let F' be the necessary sectional area and b' the admissible unit stress, for a strut subjected to loads which vary between a given maximum tension and a given maximum compression, B' being the numerically absolute maximum load, and B'' the maximum load of the opposite kind.

According to § (13), Chap. I., if there is no tendency to buckling,

$$F_1 = \frac{B_1}{b_1} = \frac{B_1}{v'.\left(1 + m_1.\dfrac{B_2}{B_1}\right)} \qquad 1$$

$$\text{and } F' = \frac{B'}{b'} = \frac{B'}{v'.\left(1 - m'.\dfrac{B''}{B'}\right)} \qquad 2$$

If there is a tendency to buckling, let l be the length of the strut, F its required sectional area, and T the mean unit stress at the moment of buckling.

$\therefore$ According to the theory of long struts,

$$T.F = \frac{E.I}{l^2} = \delta.\frac{E.I}{l^2} \qquad 3$$

δ being a co-efficient depending upon the method adopted for securing the ends, E the co-efficient of elasticity, and I the least moment of inertia of the section.

Also let t be the statical compressive strength of the material of the strut, and take $t = \mu.T$

$$\therefore \mu = \frac{t}{T} = \frac{t.F.l^2}{\delta.E.I} = \frac{F.l^2}{\sigma.I} \qquad 4$$

$$\text{where } \sigma = \frac{\delta.E}{t} \qquad 5$$

If the strut under a pressure B were not liable to buckling, its required

sectional area would be $\frac{B}{t}$, and the unit stress for an area F would be $\frac{B}{F}$.

If the strut under the pressure B be liable to buckling, its required sectional area will be $\frac{B}{T}$; let x be the unit stress, at the moment of buckling, for the area F.

Assuming that the unit stress in the two cases are in the same ratio as the required sectional areas;

$$\therefore x : \frac{B}{F} :: \frac{B}{T} : \frac{B}{t}$$

$$\therefore x = \frac{B}{F} \cdot \frac{t}{T} = \mu . \frac{B}{F} \qquad (6)$$

The force which, when uniformly distributed over the area F, will produce this stress, is $F.x = \mu.B$.

Hence, allowance may be made for buckling, by substituting for the *compressive* forces in equations (1) and (2), their values multiplied by μ. Thus, (1) becomes,

$$F = \frac{\mu.B_1}{b_1} = \frac{\mu.B_1}{v_1.\left(1 + m_1.\frac{\mu.B_2}{\mu.B_1}\right)} = \frac{\mu.B_1}{v_1\left(1 + m_1.\frac{B_2}{B_1}\right)} = \mu.F_1 \qquad (7)$$

and equation (2) becomes,

$$F = \frac{\mu.B'}{b_1} = \frac{\mu.B'}{v_1.\left(1 - m'.\frac{B''}{\mu.B'}\right)}, \text{ if } B' \text{ is a compression} \qquad (8)$$

$$\text{and } F = \frac{B'}{b'} = \frac{B'}{v'\left(1 - m'.\frac{\mu.B''}{B'}\right)}, \text{ if } B'' \text{ is a compression.} \qquad (9)$$

If $\mu < 1$, equations (1) and (2) give larger sectional areas than equations (7), (8) and (9), so that the latter are to be applied only when $\mu > 1$.

Method II.–General formulæ applicable to all values of μ may be obtained by following the same line of reasoning as that adopted in the proof of Gordon's formula. It is there assumed that the total unit stress in the most strained fibre is $p_1.\left(1 + a.\frac{l^2}{h^2}\right)$, p_1 being the stress due to direct compression, and $p_1.a.\frac{l^2}{h^2}$ that due to the bending action.

So, instead of employing equations (1) and (2) when $\mu < 1$, and equations (7), (8), and (9), when $\mu > 1$, formulæ including *all* cases

may be obtained by substituting for the compressive forces in equations (1) and (2) their values multiplied by $1+\mu$.

Thus, equation (1), becomes,

$$F=\frac{(1+\mu).B_1}{v.\left(1+m_1.\frac{B_2}{B_1}\right)}=(1+\mu).F_1 \qquad (10)$$

and equation (2) becomes,

$$F=\frac{(1+\mu).B'}{v'.\left(1-m'.\frac{B''}{(1+\mu B')}\right)}, \text{ if } B' \text{ is a compression.} \qquad (11)$$

$$\text{or } F=\frac{B'}{v'.\left(1-m'\frac{(1+\mu).B''}{B'}\right)}, \text{ if } B'' \text{ is a compression.} \qquad (12)$$

Equations (7), (8), (9), respectively, give larger values of F than the corresponding equations (10), (11), and (12).

Remarks.—For wrought-iron bars it may be assumed, as in § 11, 12, Chap. I. that $v_1=v'=700^k$ per cent². and $m_1=m'=\frac{1}{2}$.

The value of σ is given by Formula (5), but is unreliable, and varies in practice from 10,000 to 36,000 for struts with *fixed* ends.

When the ends are fixed, $\delta=4.\pi^2$, according to theory,

$$\therefore\ \sigma=4.\pi^2.\frac{E}{t}.$$

$\therefore$ if $E=2{,}000{,}000^k$ per cent². and $t=3{,}300^k$ per cent²., $\therefore \delta=23{,}926$, or in round numbers, 23,900; 24,000 is the value usually adopted by Weyrauch.

Example.—The load upon a wrought-iron column 360-centimetres long varies between a compression of $50{,}000^k$ and a compression of $25{,}000^k$; calculate the sectional area of the column, assuming it to be *first* solid and *second* hollow, allowance being made for buckling.

First:—By Eq. 1, $F_1=\dfrac{50{,}000}{700\left(1+\frac{1}{2}.\frac{25{,}000}{50{,}000}\right)}=\dfrac{400}{7}=\pi.r^2,$

r being the radius of the section.

$$\text{Also, } I=\frac{\pi.r^4}{4}, \therefore \frac{F}{I}=\frac{4}{r^2}=\frac{11}{50}$$

$$\therefore \text{By Eq. 4, } \mu=\frac{360\times 360}{24{,}000}\times\frac{11}{50}=1.188$$

$\therefore\ \mu>1$, and by Eq. 7 the required sectional area is $F_1\times 1.188$

$$=\frac{400}{7}\times 1.188=67.9 \text{ cent.}^2.$$

Second:—$F_1=\frac{400}{7}=\pi.(r_1^2-r_2^2)$, r_1 being the external and r_2 the internal radius of the section. Let $r_1=9$. cent^{rs}. and $r_2=7.92$ cent^{rs}.,

$$\therefore \pi.(r_1^2-r_2^2)=57.43\text{—cent}^2.$$

$$\text{Also } I=\pi.\frac{(r_1^4-r_2^4)}{4}, \text{ and } \therefore \frac{F}{I}=\frac{4}{r_1^2+r_2^2}=\frac{4}{143.7264}$$

$$\therefore \text{By Eq. 4, } r=\frac{360\times 360}{24,000}\times\frac{4}{143.7264}=.15$$

Hence, in the latter case, since $\mu < 1$, there is no tendency to buckling.

If the area is determined by Equation (10), its value becomes $1.15\times\frac{400}{7}=65.7$. cent^2.

Hollow cast-iron pillars with flat ends and base plates.

Thickness.	Safe load in tons of 2240 lbs. per sq. in.	
	Length=20 to 24 diars.	Length=24 to 30 diars.
$\frac{3}{4}$-in. and upwards	2	$1\frac{3}{4}$
$\frac{5}{8}$-in.	$1\frac{3}{4}$	$1\frac{1}{2}$
$\frac{1}{2}$-in.	$1\frac{1}{2}$	$1\frac{1}{4}$
$\frac{3}{8}$-in.	$1\frac{1}{4}$	1

Table shewing relative strengths of long pillars of different materials.

The strength of a pillar of unhardened cast-steel being 100.

"	"	"	"	best Staffordshire wrought-iron is	69.3.
"	"	"	"	Low moor cast-iron	39.7.
"	"	"	"	Dantzic oak	4.3
"	"	"	"	Red Deal	3.1

Brereton's table of the ultimate loads borne by fir or pine pillars of large scantling, the ends being adjusted without any special precautions.

Ratio of length to least breadth..................	10	15	20	25	30	35	40	45	50
Breaking weight in tons per sq. ft. of section..........	120	118	115	100	90	84	80	77	75

EXAMPLES.

(1).–A short pillar is subjected to a thrust of intensity p in the direction of its axis; if p_n, p'_n, and p_t, p'_t, respectively, are the normal and tangential intensities of the stress upon any two plane rectangular sections of the pillar, shew that $p_n + p'_n = p$, and $p_t = p'_t$.

(2).–A solid cast-iron pillar 9-ft. in height and 4-ins. in diameter supports a load of 55,000-lbs.; find the normal and shearing intensity of stress in a plane section inclined at 30° to the axis.

If the ends of the pillar are flat and firmly bedded, determine its breaking weight.

(3).–The pressure upon the end of a pillar varies *uniformly*; if the stress in any tranverse section is to be nowhere negative, shew that the deviation of the centre of pressure from the axis of a round or square pillar must not exceed $\frac{d}{8}$ or $\frac{a}{6}$, respectively, d being the diameter of the round and a the side of the square pillar

(4).–A cylindrical pillar 6-ins. in diameter supports a load of 400-lbs., of which the centre of gravity is $\frac{5}{8}$-in. from the axis; determine the greatest and least intensities of stress upon any transverse section of the pillar.

(5).–Compare the breaking weights of round cast-iron, wrought-iron, and mild-steel pillars, with flat and firmly bedded ends, each being 9-ft. in length and 6-ins. in diameter.

(6).–A hollow cast-iron pillar, with an external diameter of 9-ins., is to be substituted for the solid pillar in the preceding question; determine the thickness of the metal.

(7).–Determine the breaking weight of a solid round pillar, with both ends firmly secured, 10-ft. in length and 2-ins. in diameter, (1).–if of cast-iron, (2).–if of wrought-iron, (3).–if of steel (mild).

(8).–A hollow cast-iron pillar 12 ft. in height has to support a steady load of 33,000-lbs.; its internal diameter is $5\frac{1}{2}$-ins., find the thickness of the metal, the factor of safety being 6.

(9).–A solid wrought-iron pillar is to be substituted for the pillar in the preceding question; find its diameter.

(10).–What is the breaking weight of a hollow cast-iron pillar, 9-ft. in length and 6-ins. square, the metal being 1-in. thick?

(11).–Compare the breaking weight of a solid square pillar of wrought-iron, 20-ft. long and 6-ins. square, with that of a solid rectangular pillar of the same material, the section being 9 ins. by 4-ins.

(12).–Compare the breaking weights, as derived from Hodgkinson's and Gordon's formulæ, of a solid round cast-iron pillar, 20-ft. in length and 10-ins. in diameter, (1).–both ends being securely fixed, (2).–both ends being imperfectly fixed.

What load will the pillar bear, if subject to vibration?

(13).–Determine, by Hodgkinson's formula, the diameter of a solid round wrought-iron pillar equal in length and strength to that in the preceding question.

(14).–A solid or hollow pillar, of cast-iron, wrought-iron, or mild-steel, is to be designed to carry a *steady* load of 30,000-lbs.; determine the necessary diameter in each case.

(The pillar is to be 12-ft. high, and the metal of the hollow pillar is to be $\frac{3}{8}$-in. thick.)

(15).–Determine the load in the preceding question that will produce a stress of 9,000-lbs. per sq. in. section of metal.

(16).–A pillar of diameter D supports a given load; if N pillars, each of diameter d, are substituted for this single pillar, shew that d must lie between $\frac{D}{N^{\frac{1}{2}}}$ and $\frac{D}{N^{\frac{1}{4}}}$.

(17)–A solid round pillar of mild steel, 16-ft. high, supports a *steady* load of 20,000-lbs.; what is its diameter?

Find the diameter of each of 4 pillars of the same material which may be substituted for the single pillar.

(18).–What is the breaking weight of a cast-iron stanchion, of a regular cruciform section, and 15-ft. in height, the arms being 24-ins. by 1-in.?

(19).–Each of the pillars supporting the lowest floor of a refinery is $6\frac{1}{2}$-ft. high, is of a regular cruciform section, and carries a load of 240,000-lbs.; the *total* length of an arm is 26-ins., determine its thickness, the factor of safety being 10.

(20).–A compression member of a structure consists of two flat $\frac{1}{2}$-in. bars, separated by an interval of $\frac{1}{2}$-in. At what intervals should the bars be riveted together (with an $\frac{1}{2}$ in. distance piece) in order that the member may be of the same strength as a single rib of the same sectional area and depth?

(The ends are riveted in position, and the length between bearings is 6-ft.)

(21).–Compare the breaking weights of a pine timber pillar 12-ins. square and 20-ft. high, as given by Gordon's formula, and also by Rondelet's and Brereton's rules. (See Tables at end of chapter.)

(22).–Determine the breaking weight of an oak pillar 9 ft. high, 11-ins. wide, and 5-ins. thick.

P.P. (23).–What weight will be safely borne by a pillar of white deal, subject to vibration, 10-ft. high and 6-ins. square?

(24).–Determine the breaking weight of a memel pillar 16-ft. high and 13-ins. square, and compare the result with that given by Rondelet's and Brereton's rules.

(25).–Will a long pillar, if nearly straight, to begin with, deflect gradually as the load is increased, or will it give way all at once?

(26).–Two pillars are similar in all respects, except that one is straight and the other slightly bent; compare the behaviour of the pillars as they are gradually loaded until they break.

(27).–The diameter of a long pillar being unity, shew that its lateral deflection per unit of length is $\frac{1}{4}$-th of the amount of extension or compression in the outside layers per linear unit.

(28).–A long pillar of length l is deflected under a vertical load at its end; if $p.l$ is the diminution in the length of the pillar, shew that the neutral axis is displaced through a distance $p.\rho$ from the centre of the pillar, $\frac{1}{\rho}$ being the curvature and p some constant.

(29).–A pier consists of N-rows of posts, equidistant from each other; d is the distance from centre to centre of the outside rows; W is the gross vertical load of the pier; H is the greatest horizontal thrust, and acts upon the pier at a height y above the base; assuming the principle of a uniformly varying stress, the portion of the load borne by the n-th row of posts measured from the centre line is $\frac{W}{N} + a.d.\frac{2n-1}{N-1}$; find the value of the coefficient a in terms of d, H, y, and N, and determine the best value for d.

(30).–P is the vertical load of a hollow cylindrical pillar; H is a horizontal thrust which acts upon the pillar at a height y above the section at which the stress is to be calculated; d is the mean between the external and internal diameters of the pillar; A is the sectional area of the metal; shew that the greatest intensity of stress at the given section is nearly $\frac{1}{A}\cdot\left(\frac{4.H.y}{d} \pm P\right)$, the upper or lower sign being taken according as the stress i a compression or tension.

Explain the effect of making $d=\frac{4.H.y}{P}$

(31).–Prove that the flexural rigidity of a straight beam, of sectional area A, under a thrust P per unit of area, is $E.A.k^2\left(1-\frac{P}{E}\right)$, and that the beam will bend if its length, when unstrained, exceeds

$$\pi.\sqrt{\left\{\frac{E.k^2}{P} \div \left(1-\frac{P}{E}\right)\right\}}$$

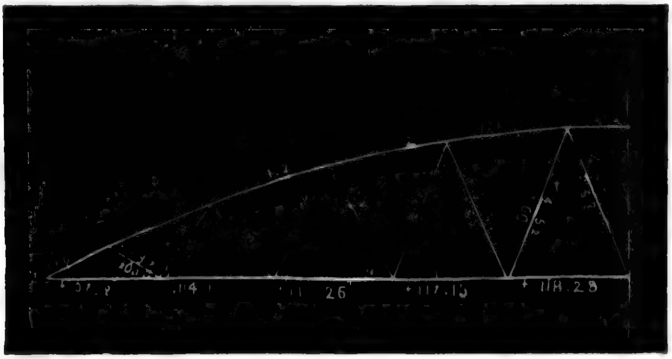
$A.k^2$ being the moment of inertia of the section, and E the coefficient of elasticity of the material.

(32).–Determine the sectional area of a double-tee strut which is to carry a load varying between a maximum tension of 80,000-lbs. and a maximum compression of 60,000-lbs. The strut is to be 6-ft. long and to consist of a 12-ins. by $\frac{3}{8}$-in. web and four equal-sided $\frac{3}{8}$-in. angle-irons.

(33).–What should be the area of the strut in the preceding question if it is made 12-ft. in length?

(34).–The web members of a Warren girder are bars of rectangular section and 10-ft. in length. One of the bars has to carry loads varying between a steady maximum tension of 20.2-tons and a maximum tension of 40.4-tons, and another to carry loads varying between a maximum compression of 8.7-tons and a maximum tension of 14.4 tons; find the sectional area in each case, allowance being made for buckling in the latter.

(35).—The diagram represents one-half of an isosceles bowstring girder 80-ft. long and 10-ft. deep at the centre; the curved flange has a



radius of 85-ft.; the permanent load is ½ ton, and the passing load 1-ton per running foot. The values of the extreme variations of stress to which any member is liable are marked on the diagram, the tensions being *positive* and the compressions negative. Determine the sectional areas of the several members.

CHAPTER C.

DEFLECTION OF STRUTS.

(1).–A pressure bar, or strut, tends to bend laterally.

The following four cases arise:–

Case I.–A strut with both ends free, or hinged, but guided in the direction of the thrust, (Fig. 1).

Case II.–A strut with one end fixed, the other being free, or hinged, but guided in the direction of the thrust, (Fig. 2).

Case III.–A strut with one end fixed, the other free, (Fig. 3).

Case IV.–A strut with both ends fixed, (Fig. 4).

(2).–***Struts with both ends round.***–The strut OA, of length l, is deflected under a pressure P, and its axis assumes the curved form OMA.

Take O as the origin, the vertical through O as the axis of x, and the horizontal through O as the axis of y.

Consider a section at any point M, (x,y).

If there is equilibrium, the equation of moments at M is,

$$- E.I.\frac{d^2y}{dx^2} = P.y \qquad 1)$$

$$\text{or, } \frac{d^2y}{dx^2} = - a^2.y, \text{ where } a^2 = \frac{P}{E.I}.$$

Multiply each side of the equation by $\frac{dy}{dx}$, and integrate,

$$\therefore \left(\frac{dy}{dx}\right)^2 = a^2(b^2-y^2),$$

b being a constant of integration.

$$\text{Hence, } \frac{dy}{\sqrt{b^2-y^2}} = a.dx,$$

$$\text{and integrating, } sin^{-1}\frac{y}{b} = ax + c,$$

c being a constant of integration.

$$\therefore\ y = b.sin(ax+c).$$

But when $x=0$, y is also 0, and $\therefore b=0$ or $c=0$.

If $b=0$, y is always 0 and lateral flexure is impossible.

$$\text{If } c=0,\ y=b.\ sin\ a.x. \tag{2}$$

Assuming that the difference of length between OMA and ONA is insensible, y is 0 when $x=l$, and $\therefore 0=b.\ sin\ al$, or, $a.l.=n.\pi$,

$$\therefore \sqrt{\frac{P}{E.I}}.l=n.\pi$$

$$\text{and } P=n^2.E.I.\frac{\pi^2}{l^2}. \tag{3}$$

Now the least value of P evidently corresponds to $n=1$, and the minimum force that will bend the strut laterally is,

$$P=E.I.\frac{\pi^2}{l^2} \tag{4}$$

Cor.–If the strut is made to pass through N points dividing the vertical OA into $N+1$ equal divisions,

$$\therefore\ y=0 \text{ when } x=\frac{l}{N+1}, \text{ and by equation (2),}$$

$$\frac{a.l}{N+1}=n.\pi, \quad \text{or} \quad P=n^2.E.I.\frac{\pi^2}{l^2}\cdot(N+1)^2.$$

As before, the least value of P corresponds to $n=1$,

$$\text{and } \therefore\ \ P=E.I.\frac{\pi^2}{l^2}.(N+1)^2,$$

is the least force that will bend the strut laterally.

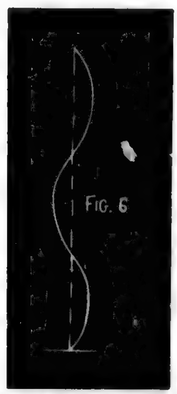


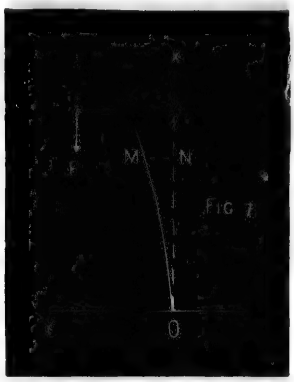
FIG. 6

Hence, the strength of the strut is increased in the ratio of 4, 9, 16,......by causing it to pass through points which divide its length into 2, 3, 4, ...equal parts, respectively.

(3).—*Struts with one end fixed and the other round, but guided in the direction of thrust.*—A strut of this class may be regarded as having about twice the strength of the strut in Case I., (§ (1), chap. IV), and the least pressure that will bend it laterally is,

$$P = 2.\, E.I.\, \frac{\pi^2}{l^2}.$$

(4).—*Struts with one end fixed and the other round.*—A rigid arm AB is connected with the free end A of a strut, and a vertical force P, applied at B, bends the strut laterally until its axis assumes the curved form OMA.



Take the same origin and axes, as before.

Let $AB = q$, $AC = p$, and let l be the length of the strut $= OC$, nearly.

If there is equilibrium, the equation of moments at any point M, (x, y), is,

$$E.I. \frac{d^2y}{dx^2} = P.(p + q - y). \qquad (1)$$

$$\text{or, } \frac{d^2y}{dx^2} = a^2(p + q - y), \text{ where, } a^2 = \frac{P}{E.I}$$

Multiply each side of the equation by $2\frac{dy}{dx}$, and integrate,

$$\therefore \left(\frac{dy}{dx}\right)^2 = a^2(b + 2.\overline{p+q}.y - y^2),$$

b being a constant of integration.

But $\frac{dy}{dx} = 0$, when $y = 0$, and $\therefore b = 0$,

$$\text{Hence, } \left(\frac{dy}{dx}\right)^2 = a^2(2.\overline{p+q}.y - y^2) \qquad (2)$$

$$\text{or, } \frac{dy}{\sqrt{2.(p+q).y - y^2}} = a.dx$$

$$\text{Integrating, } \cos^{-1} \frac{p + q - y}{p + q} = ax + c,$$

c being a constant of integration.

$$\text{or, } \frac{p + q - y}{p + q} = \cos(ax + c) \qquad (3)$$

But $y=0$, when $x=0$, and $\therefore c=0$,

$$\text{and } \frac{p+q-y}{p+q}=cos(ax). \tag{4}$$

Also, $y=p$, when $x=l$,

$$\therefore \quad \frac{q}{p+q}=cos(a.l) \tag{5}$$

If q is very small, or 0, $\frac{q}{p+q}$ may be neglected,

Hence, $0=cos\,a.l$, or, $al=n.\frac{\pi}{2}$, n being a whole odd number.

$$\therefore \; P=n^2\frac{1}{4}\cdot E.I.\frac{\pi^2}{l^2} \tag{6}$$

The least value of P evidently corresponds to $n=1$, and the least pressure that will bend the strut laterally is,

$$P=\frac{1}{4}.E.I.\frac{\pi^2}{l^2} \tag{7}$$

Cor. 1.–The deflection p, by equation (5) is,

$$p=q.\frac{1-cos\,al}{cos\,al}$$

Cor. 2.–The total compression at any point of a transverse section distant y_1 from the axis is,

$$\frac{P}{A}+\frac{M.y_1}{I}$$

Cor. 3.–Let the force applied at B be oblique, and let its vertical and horizontal co-ordinates be P and H, respectively.

The equation of moments at the point (x,y) now becomes,

$$E.I.\frac{d^2y}{dx^2}=P.(p+q-y)+H.(l-x)$$

A particular solution of this is,

$$0=P.(p+q-y')+N.\,(l-x)$$

$$\text{Let } y=y'+u, \text{ and } \frac{P}{E.I}=a^2$$

$$\therefore \; E.I.\frac{d^2u}{dx^2}=-P.u$$

$$\text{or, } \frac{d^2u}{dx^2}=-a^2.u$$

the solution of which, as before, is

$$u=b.\,sin\,(ax+c)=y-y'$$

$$\text{or, } y=p+q+\frac{H}{P}.(l-x)+b.\,sin\,(ax+c),$$

b and c being constants of integration.

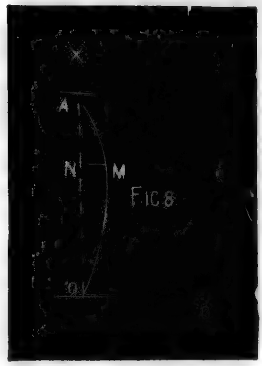
When $x=0$, y and $\frac{dy}{dx}$ are each 0; also, when $x=l$, $y=q$.

$$\therefore 0=p+q+\frac{H}{P}.l+b.\ sin\ c.$$

$$0=\frac{H}{P}+a.b.\ cos\ c.$$

$$\text{and, } q=p+q+b.\ sin\ (al+c).$$

Three equations giving b, c, and p, and therefore fully determining y.



(5).–*Strut with both ends fixed.*-Both ends of the strut being fixed, let μ be the moment of fixture at A.

The equation of moments at any point M, (x, y), of the axis is,

$$E.I.\frac{d^2y}{dx^2}=-P.y+\mu$$

$$\text{or, } \frac{d^2y}{dx^2}=a^2.(b-y),$$

$$\text{where } a^2=\frac{P}{E.I}, \text{ and } b=\frac{\mu}{P}.$$

Multiply each side of the equation by $2.\frac{d.y}{d.x}$, and integrate,

$$\therefore \left(\frac{dy}{dx}\right)^2=a^2.(2\ b.y-y^2)+d,$$

d being a constant of integration.

But $\frac{dy}{dx}=0$ when $y=0$, $\therefore d=0$

$$\text{and } \left(\frac{dy}{dx}\right)^2=a^2.(2b.y-y^2)$$

$$\text{or, } \frac{dy}{\sqrt{2.b.y-y^2}}=a.dx.$$

Integrating, $\therefore cos^{-1}\frac{b-y}{b}=ax+c$,

c being a constant of integration.

$$\text{or } \frac{b-y}{b}=cos(ax+c).$$

But $y=0$, when $x=0$, and also when $x=l$

$\therefore 1=cos\ c$, and $1=cos\ (al+c)$

$\therefore c=0$, and $a.l=2.n.\pi$, n being a whole number.

Hence, $P=n^2.4.\ E.I.\frac{\pi^2}{l^2}$

The least value of P corresponds to $n=1$, and the minimum pressure that will bend the strut laterally is,

$$P=4.E.I.\frac{\pi^2}{l^2}.$$

(6).–*Applications.*– $\dfrac{I}{A}$ is the square of the radius of gyration, $=k^2$, suppose.

Thus, the equations, giving the least value of P in the four cases, may be written,

$$\frac{P}{A}=E.k^2.\frac{\pi^2}{l^2} \qquad (a)$$

$$\frac{P}{A}=2.\,E.k^2.\frac{\pi^2}{l^2} \qquad (b)$$

$$\frac{P}{A}=\tfrac{1}{4}.E.k^2.\frac{\pi^2}{l^2} \qquad (c)$$

$$\frac{P}{A}=4\ E.k^2.\frac{\pi^2}{l^2} \qquad (d)$$

These formulæ are very easy of application, but Hodgkinson's experiments shew that the value of P, as derived from them, is rather too large, which may be explained by the fact that in the analysis the elasticity of the strut is assumed to be perfect.

The safe pressure upon a strut is $m.P$, m being $\frac{1}{5}$ for wrought-iron, $\frac{1}{6}$ for cast-iron, and from $\frac{1}{4}$ to $\frac{1}{10}$ for timber.

The formulæ are to be used only when the ratio of $\dfrac{l}{d}$ or $\dfrac{l}{2.r}$, (d being the shortest side of a rectangular section, and r the radius of a circular section), exceeds the values given in the following table :–

Material.	Value of $\frac{l}{d}$.	Value of $\frac{l}{2r}$	Formula.
Wrought-iron	28	24	Equation (a)
Cast-iron	$11\frac{1}{2}$	10	
Timber	$13\frac{1}{2}$	$11\frac{1}{2}$	
Wrought-iron	38	33	" (b)
Cast-iron	16	14	
Timber	19	16	
Wrought-iron	14	12	" (c)
Cast-iron	$5\frac{3}{4}$	5	
Timber	8	6	
Wrought-iron	56	48	" (d)
Cast-iron	23	20	
Timber	27	23	

In practice, preference is given to the empirical formulæ discussed in Chap. IV, but in reality, they are of the same form as those deduced above.

For $k^2 = \frac{d^2}{12}$ in a rectangular section, and $= \frac{r^2}{4}$ in a circular section,

$$\text{so that } \frac{P}{A} \text{ in equations } (a) \text{ to } (d), \propto \frac{d^2}{l^2} \text{ or } \propto \frac{r^2}{l^2},$$

$$\text{and } \therefore P \propto \frac{d^4}{l^2} \text{ or } \propto \frac{r^4}{l^2}.$$

(7).–*Practical Remarks.*–Many engineers object to the use of flat bars as compression members, and substitute for them bars of **T**, **L**, Barlow-rail, and other sections.

These are doubtless very efficient, but the objection to flat bars seems to have been over-estimated.

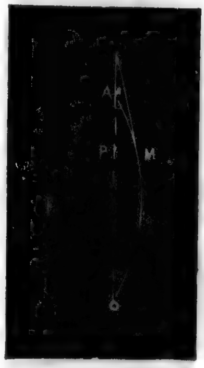
Consider the case of a flat bar hinged, or rounded, at both ends.

The bar will not bend laterally under pressure so long as the unit-stress $< E.k^2.\frac{\pi^2}{l^2}$. Let the unit-stress $=$ 8000-lbs. per sq. in., $\therefore$ if $E = 25{,}000{,}000$ lbs.,

$$8000 < 25{,}000{,}000.\frac{\pi^2}{12}.\frac{d^2}{l^2}, \text{ or } \frac{l}{d} < 50.7$$

Hence, the length of a flat bar in compression seems to be comparatively limited. If, however, both ends are securely *fixed*, the strength is *quadrupled*, and the admissible length of bar is *doubled*, while it may be still further increased by fixing the bar at intermediate points as indicated in the corollary of § (2).

(8).–*Determination of the constant b in case I.*



Let ds be the length of an element of the bent strut at M, (x, y).

Let θ be the inclination of the tangent at M to the vertical.

The pressure upon $ds = P.\cos\theta. = P.\frac{dx}{ds}$

$\therefore$ the amount by which ds is compressed

$$= \frac{P.\frac{dx}{ds}}{E.A}.ds = \frac{P}{E.A}.dx,$$

A being the sectional area of the element.

Hence, the total diminution in the length of the strut

$$=\int_0^l \frac{P}{E.A}\cdot dx = \frac{P}{E.A}\cdot l$$

Again the total length of the bent strut $=\int_0^l \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx$

$$=\int_0^l (1+ .a^2b^2.cos^2.ax)^{\frac{1}{2}} dx$$

an elliptic integral admitting of no finite solution. Approximately, however, this expression becomes,

$$\int_0^l (1+\tfrac{1}{2}.a^2.cos^2.ax).dx = \int_0^l \left(1+\frac{a^2.b^2}{4}\cdot\overline{1+cos.2ax}\right)\cdot dx.$$

$$=l+\frac{a^2.b^2}{4}\cdot\left(l+\frac{sin.2al}{2a}\right)=l+\frac{a^2.b^2}{2}\cdot l, \text{ nearly.}$$

Let L be the primitive length of the strut,

$$\therefore\ L-\frac{P.l}{E.A}=l+\frac{a^2.b^2}{2}\cdot l = l+\tfrac{1}{2}\frac{P.l}{E.I}b^2,$$

$$\text{and } b^2=2.\frac{E.I}{P}\cdot\left(\frac{L-l}{l}\right)-2\cdot\frac{I}{A}$$

(9)–*Remarks.*–More correct solutions of the four cases may be obtained by writing

$$\frac{\dfrac{d^2y}{dx^2}}{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}} \quad \text{instead of } \frac{d\,y}{dx^2} \text{ in the equation of moments.}$$

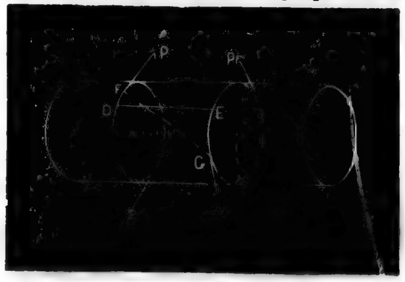
The first integral of the left hand side of the equation of moments in each case then becomes $\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}$, and the resulting equation takes the form of an elliptic integral.

It must be remembered, however, that the theoretical deductions depend upon assumptions which are only approximately true.

CHAPTER V.

TORSION.

(1).—Torsion is the force with which a thread, wire, or prismatic bar tends to recover its original state after having been twisted, and is produced when the external forces which act upon the bar are reducible to two equal and opposite couples (the ends of the bar being free), or to a single couple (one end of the bar being fixed), in planes perpendicular to the axis of the bar.



The effect upon the bar is to make any transverse section turn through an angle in its own plane, and to cause originally straight fibres, as DE, to assume helicoidal forms, as FG or DC. This induces longitudinal stresses in the fibres, and transverse sections become warped. It is found sufficiently accurate, however, in the case of cylindrical and regular polygonal prisms, to assume that a tranverse section which is plane before twisting, remains plane while being twisted.

FIG 2

In order that the bar may not be *bent*, its axis must coincide with the axis of the twisting couple.

(2).–*Coulomb's Laws.*–The angle turned through by one transverse section relatively to another at a unit distance from it is called the *Angle of Torsion*, and Coulomb deduced from experiments upon wires, that this angle is *directly* proportional to the moment of the twisting couple, and *inversely* proportional to the fourth power of the diameter.

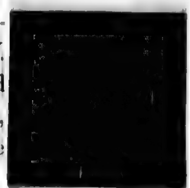
Thus, if a force P, at the end of a lever of radius p, twists a cylindrical bar of length L and radius R, and if θ is the *circular measure* of the angle of torsion,

$$\therefore \theta \propto P.p, \text{ and also } \propto \frac{1}{R^4},$$

so that $\theta = C.\dfrac{P.p}{R^4}$, C being a constant depending only upon the nature of the material.

Let T be the *total* angle of torsion, *i.e.*, the angle turned through by one end of the bar relatively to the other,

$$\therefore \frac{T}{L} = \theta = C.\frac{P.p}{R^4}$$



(3).- *Torsional strength of shafts.*—Consider a portion of the shaft bounded by the planes CE and MN. It is kept in equilibrium by the couple $(P,-P)$, and by the elastic resistance at the section MN. Hence, this elastic resistance must be equivalent to a couple equal and opposite to $(P,-P)$.

Let Fig. 3 be the transverse section at MN, and let $abb'a'$ be any elementary area $(=\Delta A_1)$ of the surface bounded by the radii OA, OB, and by the concentric arcs aa', bb'.

Let x_1 be the distance of ΔA_1 from O.

It is assumed, and is approximately true, that the resistance of any element $abb'a'$ to torsion, is directly proportional to the angle of torsion (θ), to its distance from the axis (x_1), and to its area (ΔA_1), and also that it acts at right-angles to the radial line of the element, *i.e.*, OA or OB.

Thus, the resistance of $abb'a'$ to torsion $= m.\theta.x_1.\ \Delta A_1$,

m being a constant to be determined by experiment.

The corresponding moment of resistance about the axis $= m.\theta.x_1^2\ \Delta A_1$,

Similarly, if x_2, x_3, x_4,......are the distances from the axis of any other elements, ΔA_2, ΔA_3, ΔA_4 ......, respectively, the corresponding moments of resistance are $m.\theta.x_2^2.\Delta A_2$, $m.\theta.x_3^2.\Delta A_3$,......

Hence, the *total* moment of resistance of the section,

$$= m.\theta.\,(x_1^2.\Delta A_1 + x_2^2.\Delta A_2 + x_3^2.\Delta A_3 + \ldots\ldots)$$

$$= m.\theta.\Sigma(x^2.\Delta A) = m.\theta.I,$$

I being the moment of inertia with respect to the axis.

But this moment of resistance (M) is equal and opposite to the moment of the couple $(P,-P)$.

$$\therefore M = m.\theta.I = P.p.$$

Cor. 1.–Let f be the stress *at* the point farthest from the axis. For a *solid round* shaft, of diameter D,

$$I=\frac{\pi.D^4}{32}, \text{ and } f=m.\theta.\frac{D}{2}.$$

$$\therefore M=P.p=\frac{\pi}{16}.f.D^3=\cdot 196.f.D^3$$

For a *hollow round* shaft, D being the external and D_1 the internal diameter.

$$I=\frac{\pi}{32}.(D^4-D_1^4), \text{ and } f=m.\theta.\frac{D}{2}$$

$$\therefore M=P.p=\frac{\pi}{16}.f.\frac{D^4-D_1^4}{D}=.196.f.\frac{D^4-D_1^4}{D}.$$

If the thickness (T) of the hollow shaft is small compared with D,

$\therefore D^4-D_1^4=D^4-(D-2.T)^4=8.D^3.T$, approximately,

and $M=P.p=1.57.D^2.T.$

For a *solid square* shaft, H being the side of the square, $I=\frac{H^4}{6}$, and $f=$stress at V

$$=m.\theta.OV=m\theta.\frac{H}{\sqrt{2}}.$$

$$\therefore M=P.p=\frac{f.\sqrt{2}}{H}.\frac{H^4}{6}$$

$$=\frac{\sqrt{2}}{6}f.H^3=.236.f.H^3.$$

$$\text{and } \theta=\frac{M}{m.I}=\frac{6.M}{m.H^4}.$$

In these results it is assumed that $m.\theta.,\left(\frac{2f}{D}\text{ or }\frac{\sqrt{2}.f}{H}\right)$, is constant at at different points of the cross-section, which, however, is only true for circular sections. According to St. Venant, the more correct expression for the moment of resistance of a square shaft is $P.p=\cdot 281.f.H^3$

Cor. 2.–The torsional stress per unit of area at a distance x from the axis is $m.\theta.x$.

Hence, if $\theta=1$ and $x=1$, m is the force that will twist a unit of area at a unit of distance from the axis through an angle unity.

Cor. 3.–For a solid cylinder, $P.p=\frac{m.\pi.\theta.R^4}{2}$, R being the radius.

Comparing this with the formula in § (2), it appears that $C=\frac{2}{m.\pi}$

Cor. 4.–In any transverse section of a solid cylindrical shaft, the maximum unit stress is $m.\theta.\frac{D}{2}=\frac{16}{\pi}.\frac{P.p}{D^3}=f.$

This relation is true so long as the stress does not exceed the limit of elasticity, and agrees with the *practical* rule, that *the diameter* of a cylindrical shaft subjected to torsional forces, is proportional to the cube root of the twisting couple.

The rule is usually expressed in the form, $P.p=K.D^3$,

$$\text{so that } K=\frac{f.\pi}{16}.$$

Note.–If P be the *torsional breaking* weight, *K* is called the *coefficient* of *torsional rupture.*

Cor. 5.–Let *W* be the work transmitted to the shaft in ft. lbs. per minute, and let *N* be the number of revolutions of the shaft per minute.

$$\therefore\ W=P.2\pi.p.N=2.\pi.N.P.p=2\ \pi.N.\ K\ D^3$$

$$\therefore\ \frac{W}{N}=2.\pi.K.\ D^3$$

Cor. 6.–The *Resilience* of a cylindrical axle is the product of one-half of the greatest moment of torsion into the corresponding angle of torsion.

Cor. 7.–It often happens in practice, that a shaft (or beam) is subjected to a bending as well as to a torsional action. In such a case the strength and dimensions of the shaft may be determined by calculating the moment of resistance for any section from the approximate formula,

$$M=\tfrac{3}{8}\ M_b+\tfrac{5}{8}\ (M_b^2+M_t^2)^{\frac{1}{2}}.$$

M_b being the bending moment, and M_t the torsional moment at the given section.

Cor. 8.–Cauchy found analytically that in an *isotropic* body *m* is the $\frac{2}{5}$ths the coefficient of direct elasticity.

Experiments indicate that *m* is about $\frac{3}{8}$ths or $\frac{1}{3}$rd of the coefficient of direct elasticity.

It is advisable that θ should not exceed $\frac{1^\circ}{13}$ per lineal ft.

Example.—A wrought-iron shaft in a rolling mill makes 95 revolutions per minute and transmits 120*H.P.*, which is supplied from a waterfall by means of a turbine; determine the diameter of the shaft, (1) if the maximum stress in the metal does not exceed 9000-lbs per sq. in., (2) if the angle of torsion is not to exceed $\frac{1^\circ}{13}$ per lineal ft.

As a matter of fact, the diameter of the shaft is $3\frac{3}{4}$-ins. at the bearings and 4-ins. in the intermediate lengths; what are the corresponding maximum inch-stresses in the metal?

Let the twisting couple be represented by a force P at the end of an arm p.

$$\therefore\ P.2\pi.p.95 = 120 \times 33{,}000\text{-ft. lbs.}$$

$$\therefore\ P.p. = \frac{120 \times 33{,}000}{2.\pi.95} = \frac{126{,}000}{19}\text{ ft.-lbs} = \frac{126{,}000 \times 12}{19}\text{-inch-lbs.}$$

First, $$\frac{126{,}000 \times 12}{19} = P.p = \frac{f.\pi.D^3}{16} = \frac{9000 \times 22}{16 \times 7}\cdot D^3$$

$$\therefore\ D^3 = \frac{9408}{209},$$

and diar. of shaft $= 3.56$-ins.

Second, $$\frac{126{,}000 \times 12}{19} = P.p = \frac{m.\theta.\pi.D^4}{32}$$

But $\theta = \frac{\pi}{180}\cdot\frac{1}{13} \times \frac{1}{12}$; take $m = 8{,}500{,}000$-lbs.

$$\therefore\ \frac{126{,}000}{19} \times 12 = \frac{8{,}500{,}000}{32}\cdot\frac{22}{7}\cdot\frac{1}{180}\cdot\frac{1}{13}\cdot\frac{1}{12}\cdot\frac{22}{7}.D^4$$

$\therefore\ D^4 = 723.93$, and diar. of shaft $= 5.18$-ins.

Third, the maximum stresses in the real shaft at the bearings and in the intermediate lengths, are respectively given by,

$$\frac{126{,}000}{19} \times 12 = \frac{\text{stress}}{16}\cdot\frac{22}{7}\cdot\left(3\tfrac{3}{4}\right)^3, \text{ and } \frac{126{,}000}{19} \times 12 = \frac{\text{stress}}{16}\cdot\frac{22}{7}\cdot\left(4\right)^3;$$

from the former, the maximum stress $= 7682$-lbs. per sq. in.

" " latter, " " " $= 6330$-lbs. " "

TABLES.

* Table of the values in lbs. of the coefficient m.

Cast-iron	2,850,000	Cedar, Red	890,000
Wrought-iron	8,500,000 to 10,000,000	Chestnut	355,000
		Hickory	910,000
		Locust	1,225,000
Steel	8,500,000	Mahogany	660,000
Forged Steel	14,230,000	Oak	570,000
Brass, wire-drawn	5,330,000	Pine, Spruce	211,000
Bronze	1,516,000	" Yellow	495,000
Copper	6,200,000	" White	220,000
Ash	410,000	Walnut, Black	582,000

* Table of the safe working values in lbs. of f and k, in the formulæ

$$P.p = f.\tfrac{\pi}{16}.D^3 = K.D^3.$$

MATERIAL.	f	K	MATERIAL.	f	K
Cast-iron	5000	982	Hickory	820	80.5
Wrought-iron	9000	1768	Locust	825	81
Puddled Steel	8500	1670	Oak	540	53
Cast-steel	12,000 to 18,000	2357 to 3535	Mahogany, Spanish	667	65.5
			Black Spruce	275	27
			Heart-wood	336	33
Ash	417	41	Sap-wood	402	39.5
Cedar, Red	310	30.5	Pine, White	235	23.1
Chestnut	376	37	Walnut, Black	524	51.6

† Table of the values in lbs. of the coefficient of torsional rupture.

Cast-iron	5,400	Elm	274
Wrought-iron	9,800	Larch	190 to 333
Steel, Bessemer	15,000	Oak	451
Steel, Crucible, hammered	17,000	Red Pine	98 to 157
Ash	274	Spruce Fir	118

* The values for the different timbers are those obtained by Prof. Thurston with his recording apparatus.

† Theory of Strains.—Stoney.

EXAMPLES.

(1).–A steel shaft 4-ins. in diar. is subjected to a twisting couple which produces a circumferential stress of 15,000-lbs.; what is the stress (shear) at a point 1-in. from the centre of the shaft?

Determine the twisting couple.

(2).–A round cast-iron shaft, 15-ft. in length, is acted upon by a weight of 2,000-lbs. applied at the circumference of a wheel on the shaft; the diar. of the wheel is 2-ft., find the diar. of the shaft so that the total angle of torsion may not exceed 2°.

(3).–A wrought-iron shaft is subjected to a twisting couple of 12,000-ft.-lbs.; the length of the shaft between the sections at which the power is received and given off is 30-ft.; the total admissible twist is 4°; find the diar. of the shaft.

(4).–A crane chain exerts a pull of 6000-lbs. tangentially to the drum upon which it is wrapped; find the diar. of a wrought-iron axle which will transmit the resulting couple, the *effective* radius of the drum being $7\frac{1}{2}$-ins.

(5).–A turbine makes 114 revolutions per minute and transmits 92 *H.P.* through the medium of a shaft 8-ft.-6-ins. in length; what must be the diar. of the shaft so that the total angle of torsion may not exceed $\frac{2}{3}^{\circ}$.

Determine the side of a square pine shaft that might be substituted for the iron shaft.

(6).–A steel shaft 20-ft. in length and 3-ins. in diar. makes 200 revolutions per minute and transmits 50-*H.P.*; through what angle is the shaft twisted?

A wrought-iron shaft of the same length is to do the same work at the same speed; find its diar. so that the stress at the circumference may not exceed $\frac{3}{5}$-ths of that at the circumference of the steel shaft.

(7).–A vertical cast-iron axle in the Saltaire works makes 92 revolutions per minute and transmits 300 *H.P.*; its diar. is 10-ins., find the the angle of torsion.

(8).–In a spining-mill, a cast-iron shaft $8\frac{1}{4}$-ins. in diar. makes 27 revolutions per minute; the angle of torsion is not to exceed $\frac{1}{13}^{\circ}$ per lineal ft.; find the work transmitted.

(9).–A square wooden shaft 8-ft. in length is acted upon by a force of 200-lbs. applied at the circumference of an 8-ft. wheel on the shaft; find the length of the side of the shaft so that the total torsion may not exceed 2°.

What should be the diar. of a round shaft of equal strength and of the same material?

(10).–The crank of a horizontal engine is 3-ft. 6-ins. and the connecting rod 9-ft. long. At half-stroke the pressure in the connecting rod is 500-lbs.; what is the corresponding twisting moment in the crank-shaft?

(11).–If the horizontal pressure upon the piston end of the connecting rod in the preceding question is constant, find the *maximum* twisting moment in the crank-shaft.

(12).–The wrought-iron screw-shaft of a steamship is driven by a pair of cranks set at right angles and 21.7 ins in length; the horizontal pull upon each crank pin is 176,400-lbs., and the effective length of the shaft is 866-ins.; find the diar. of the shaft so that, (1).–the circumferential stress may not exceed 9000-lbs. per sq. in. (2).–the angle of torsion may not exceed $\frac{1^\circ}{13}$ per lineal ft.

The *actual* diar. of the shaft is 14.9-ins., what is the *actual* torsion?

(13).–What is the torsion in the preceding question, when one of the cranks passes a dead point?

(14).–An iron shaft 300-ft. in length makes 200 revolutions per minute and transmits 10 *H.P.*; determine its diar so that the greatest stress in the material may be the same as the stress at the circumference of an iron shaft 1-in. in diar. and transmitting 500 ft. lbs.

(15).–Determine the *coefficient of torsional rupture* for the shaft in Question (12), 10 being the factor of safety.

(16).–A wrought-iron shaft in a rolling mill is 220-ft. in length, makes 95 revolutions per minute and transmits 120 *H.P.* to the rolls; the main body of the shaft is 4-ins in diar., and it revolves in gudgeons $3\frac{3}{4}$-ins. in diar.; find the greatest shear stress in the shaft proper and in the portion of the shaft at the gudgeons.

(17).–Power is taken from a shaft by means of a pulley 24-ins. in diar., which is keyed on to the shaft at a point dividing the distance between two consecutive supports into segments of 20 and 80-ins.; the tangential force at the circumference of the pulley is 5500-lbs.; if the shaft is of cast-iron, determine its diar., taking into account the bending action to which it is subjected.

(18).–If a round bar of any material is subjected to a twisting couple, shew that its maximum resilience is $\frac{2}{3}$-rds. of the maximum resilience of the material.

(19).–Determine the diameter of a wrought-iron shaft for a screw steamer; the indicated $H.P = 1000$, the number of revolutions per minute $= 150$, and the length of shaft from thrust bearing to screw $= 75$-ft.

(20).–In a spinning mill, a cast-iron shaft, 84-ft. long, makes 50 revolutions per minute and transmits 270 *H.P.*, find its diameter, (1).–if the stress in the metal is not to exceed 5,000-lbs. per sq. in., (2).–if the angle of torsion per lineal foot is not to exceed $\frac{1^\circ}{13}$.

Also, in the first case find the total torsion.

CHAPTER VI.

STRENGTH OF HOLLOW CYLINDERS AND SPHERES.

(1).–*Thin hollow cylinders; boilers; pipes.*–

Let r be the radius of the cylinder.

Let t be the thickness of the metal.

Let p be the fluid pressure upon each unit of surface.

Let f be the tensile or compressive unit stress according, as p is an internal or external pressure.

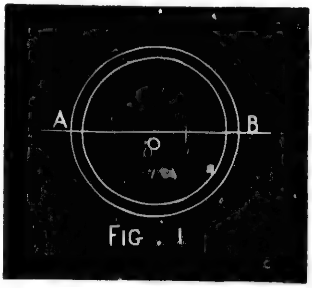


FIG. 1

Fig. 1 represents a cross-section of the cylinder of thickness *unity*.

A section made by any diametral plane, as AB, must develop a total resistance of $2.t.f$, and this must be equal and opposite to the resultant of the fluid pressure upon each half, *i.e.*, to $p.2r$.

$$\text{Hence, } 2.t.f = 2.p.r, \text{ or } t.f = p.r. \tag{1}$$

The total pressure upon each of the *flat* ends of the cylinder$=\pi.r^2.p$.

$$\therefore \text{ the } longitudinal \text{ tension in a thin hollow cylinder} = \frac{\pi.r^2 p}{2\text{-}\pi.r.t} = \frac{p.r}{2.t}. \tag{2}$$

and is one-half of the circumferential stress f.

Cor. 1.–Let the cylinder be subjected to an external pressure p' as well as to an internal pressure p.

$$\therefore f.t = p.r - p'.r', \tag{3}$$

r' being the radius of the outside surface of the cylinder.

f is a tension or pressure according as $p.r. \gtrless p'.r'$.

Generally, $r - r'$ is very small, and the relation (3) may be written,

$$f.t = r.\,(p - p')$$

(2).–*Thick hollow cylinder.*—Equations 1 to 4 are only approximate and depend upon the assumption that the thickness t is small as compared with the radius r, and that the stress in the metal is uniformly distributed over the thickness. Also, it is the inner, or maximum circumferential stress that is limited by the strength of the metal, while equations 1 and 3 give the *mean* circumferential stress only. Suppose the annulus forming the section of a cylinder to be composed of an indefinite number of concentric rings. Let dx be the thickness of one of the rings which exerts a tension of intensity dq, and let x be its distance from the axis.

$\therefore p.r - p'.r' =$ difference between the total pressures from within and without $=$ total circumferential stress $= \int_r^{r'} dq$

If it be assumed that the thickness $(= r' - r)$ remains unchanged under the pressure, then the circumferential extension of each of the concentric rings must be equal to the same constant quantity λ.

$$\therefore dq = E.dx \cdot \frac{\lambda}{2.\pi.x},$$

E being the coefficient of elasticity.

$$\text{and } \therefore p.r - p'.r' = \frac{E.\lambda}{2.\pi} \int_r^{r'} \frac{dx}{x} = \frac{E.\lambda}{2.\pi} \cdot \log_e \frac{r'}{r}$$

Let f be the bursting, proof, or working tensile unit-stress, $\therefore f = E.\frac{\lambda}{2.\pi.r}$

$$\therefore p.r - p'.r' = f.r.\log_e \frac{r'}{r}.$$

$$\text{or } \frac{r'}{r} = e^{\frac{p.r - p'.r'}{f.r}}$$

$$\therefore \frac{r'}{r} = 1 + \frac{p.r - p'.r'}{f.r} + \frac{1}{2} \cdot \left(\frac{p.r - p'.r'}{f.r}\right)^2, \text{ approximately.}$$

if p is small as compared with f,

$$\text{and } \therefore \frac{t}{r} = \frac{r'}{r} - 1 = \frac{p.r - p'.r'}{f.r} \cdot \left(1 + \frac{p.r - p'.r'}{2.f.r}\right) \qquad (8)$$

In most cases which occur in practice p' is so small as compared with p that it may be disregarded.

Hence, making p' zero in equation (8),

$$\frac{t}{r} = \frac{p}{f} \cdot \left(1 + \frac{1}{2}\frac{p}{f}\right) \qquad (9)$$

Cor.—Rankine, in his App. Mechs., obtains by another method,

$$\frac{r'}{r} = \sqrt{\frac{f + p}{f - p + 2.p'}} = \sqrt{\frac{f + p}{f - p}},$$

if p' be neglected.

$$\therefore \frac{r'}{r} = \left(1 + \frac{p}{f}\right)^{\frac{1}{2}} \left(1 - \frac{p}{f}\right)^{\frac{1}{2}} = \left(1 + \frac{1}{2}\frac{p}{f} - \frac{1}{8}\frac{p^2}{f^2}\right) \cdot \left(1 + \frac{1}{2}\frac{p}{f} + \frac{3}{8}\frac{p^2}{f^2}\right)$$

$$= 1 + \frac{p}{f} + \frac{1}{2}\frac{p^2}{f^2}, \text{ approximately,}$$

if p is small as compared with f.

$$\therefore \frac{r'}{r} - 1 = \frac{t}{r} = \frac{p}{f} \cdot \left(1 + \frac{1}{2}\frac{p}{f}\right), \text{ an equation identical with (9).}$$

(3).–*Spherical Shells.*–Let the data be the same as before.

The section made by any diametral plane must develop a total resistance of $2.\pi.r.t.f$,

$$\therefore\ 2\pi.r.t.f=\pi.r^2.p, \text{ or } 2.t.f=p.r. \qquad (5)$$

Hence, a spherical shell is *twice* as strong as cylindrical shell of the same diameter and thickness of metal, so that the strongest parts of *egg-ended* boilers are the ends.

Cor.(1).–Let the shell be subjected to an external pressure p' as well to an internal pressure p.

$$\therefore\ 2\pi.\frac{r'+r}{2}.t.f=\pi.r^2.p-\pi r'^2.p'$$

$$\therefore\ f.(r'+r).t=r^2.p-r'^2.p'. \qquad (6)$$

f is a tension or pressure according as $r^2.p \gtrless r'^2.p'$

Generally $r'-r$ is very small, and the relation (6) may be written,

$$f.t=\frac{r}{2}.(p-p')$$

Cor.(2).–For a thick hollow sphere, Rankine obtains,

$$p=f.\frac{2.r'^3-2.r^3}{r'^3+2.r^3}, \text{ approximately.}$$

(4).–*Practical Remarks.*—A common rule requires that the working pressure in fresh-water boilers should not exceed $\frac{1}{6}$th of the bursting pressure, and in the case of marine boilers that it should not exceed $\frac{1}{7}$th.

An English Board of Trade rule is that the tensile working stress in the boiler plate is not to exceed 6,000-lbs. per sq. in. of gross section, and French law fixes this limit at 4250-lbs. per sq. in.

The thickness to be given to the wrought-iron plates of a cylindrical boiler is,

according to French law, $t=.0036.n.r+.1$-in.

according to Prussian law, $t=\left(e^{.003n}-1\right)r+.1\text{-in}=.003.n.r+.1$-in, *app.y*, r being the radius in inches, and n the excess of the internal above the external pressure in atmospheres.

The thickness given to cast-iron cylindrical boiler tubes is,

according to French law, 5 times the thickness of equivalent wrought-iron tubes,

according to Prussian law, $t=\left(e^{.01n}-1\right).r+\frac{1}{3}\text{-in.}=.01.n.r+\frac{1}{3}$-in., *app.y*.

Steam boilers before being used should be subjected to a hydrostatic test varying from $1\frac{1}{2}$ to 3-times the pressure at which they are to be worked.

The following formula, deduced by Fairbairn from an extensive series of experiments, gives the collapsing pressure of a wrought-iron cylindrical tube, or flue; the collapsing pressure in lbs. per sq. in. of surface $= p = 403,150.\frac{t^{2.19}}{l.r}$, t, r, being the thickness and radius in inches and l the length in ft.

In practice, however, t^2 may be generally used instead of $t^{2.19}$.

The experiments also shewed that the strength of an elliptical tube is almost the same as that of a circular tube of which the radius is the radius of curvature at the ends of the minor axis. Hence, if a and b are the major and minor axes of the ellipse, the above formula becomes,

$$p = 403,150.\frac{b}{a^2}.\frac{t^{2.19}}{l.r.}$$

The thickness of tubes subjected to external pressure is,

according to French law, twice the thickness of tubes subjected to interior pressure, but under otherwise similar conditions;

according to Prussian law the thickness of heating pipes is,

$t = .0067d.\sqrt[3]{n} + .05$-in., if of sheet-iron.

and $t = .01\ d\sqrt[3]{n} + .07$-in., if of brass.

EXAMPLES.

(1).–What should be the thickness of the plates of a cylindrical boiler, 6-ft. in diar., and worked to a pressure of 50-lbs. per sq. in., in order that the working tensile stress may not exceed 1.67-tons per sq. in. of gross section ?

(2).–A cylindrical boiler with hemispherical ends is 4-ft in diar., and 22-ft. in length; determine the thickness of the plates for a steam pressure of 4 atmospheres.

(3).–What is the collapsing pressure of a flue 10-ft. long, 36-ins in diar., and composed of $\frac{1}{2}$ in plates ?

(4).–Determine the thickness of a 2-in. locomotive fire tube to support an external pressure of 5-atmospheres.

(5).–A thin, hollow, spherical, elastic envelope, whose internal radius is R, was subjected to a fluid pressure, which caused it to expand gradually until its radius became R_1; determine the work done.

(6).–The plates of a cylindrical boiler, 5-ft. in diameter are $\frac{1}{2}$-in thick; find to what pressure the boiler may be worked so that the tensile stress in the plates may not exceed $1\frac{1}{2}$-tons per sq. in. of gross section.

(7).–Shew that the assumption of a uniform distribution of stress in the thickness of a cylindrical or spherical boiler is only admissible when the thickness is very small.

(8).–Assuming that the annulus forming the section of a cylindrical boiler is composed of a number of infinitely thin rings, shew that the pressure at the circumference of a ring of radius r is $\frac{A}{r^{1+m}}$ per unit of surface, and that the circumferential stress is $\frac{B}{r} + \frac{A}{m.r^{m+1}}$, A and B denoting arbitrary constants, and m the coefficient of lateral contraction.

Find the values of A and B, p_o and p_1, being respectively, the internal and external pressures.

(9).–Shew that in the case of a spherical boiler, the pressure and circumferential stress are respectively $\frac{A}{r^{1+m}}$ and $\frac{B}{r^2} + \frac{2.A}{(m-1).r^{m-3}}$.

Find A and B.

THE END.